A TREATISE ON

PLANE CO-ORDINATE GEOMETRY

AS APPLIED TO THE STRAIGHT LINE

AND THE

CONIC SECTIONS.

With Numerous Examples.

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PREFACE.

I HAVE endeavoured in the following Treatise to exhibit the subject in a simple manner for the benefit of beginners, and at the same time to include in one volume all that students usually require. In addition, therefore, to the propositions which have always appeared in such treatises, I have introduced the methods of *abridged notation*, which are of more recent origin; these methods which are of a less elementary character than the rest of the work, are placed in separate chapters, and may be omitted by the student at first.

The examples at the end of each chapter, will, it is hoped, furnish sufficient exercise on the principles of the subject, as they have been carefully selected with the view of illustrating the most important points, and have been tested by repeated experience with pupils. At the end of the volume will be found the results of the examples, together with hints for the solution of some which appear difficult.

The properties of the parabola, ellipse, and hyperbola, have been separately considered before the discussion of the general equation of the second degree, from the belief that the subject is thus presented in its most accessible form to students in the early stages of their progress.

I. TODHUNTER.

ST JOHN'S COLLEGE,
July, 1855.

In the second edition the work has been revised and some additions have been made both to the text and to the examples; the hints for the solution of the examples have also been considerably increased.

March, 1858.

In the third edition some articles which experience proved to be difficult for students have been simplified and improved, and a few additional illustrations have been introduced. In consequence of the demand for the work proving much greater than had been originally anticipated, a large number of copies has been printed, and a considerable reduction effected in the price.

January, 1862.

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PLANE CO-ORDINATE GEOMETRY.

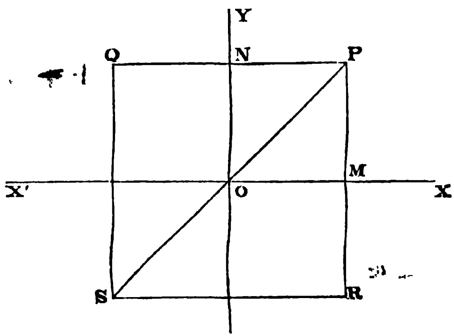
Students reading this work for the first time may omit Chapters IV, VII, XIV, XV, XVI.

PLANE CO-ORDINATE GEOMETRY.

CHAPTER I.

CO-ORDINATES OF A POINT.

1. In Plane Co-ordinate Geometry we investigate the properties of straight lines and curves lying in one plane by means of *co-ordinates;* we commence by explaining what we mean by the *co-ordinates of a point.*



Let O be a fixed point in a plane through which the lines $X'OX$, $Y'OY$, are drawn at right angles. Let P be any other point in the plane; draw PM parallel to OY meeting OX in M, and PN parallel to OX meeting OY in N. The position of P is evidently known if OM and ON are known; for if through N and M lines be drawn parallel to OX and OY respectively, they will intersect in P.

The point O is called the *origin;* the lines OX and OY are called *axes;* OM is called the *abscissa* of the point P; and

ON, or its equal MP, is called the *ordinate* of P. Also OM and MP are together called *co-ordinates* of P.

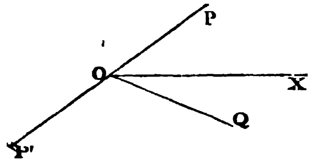
2. Let $OM=a$, and $ON=b$, then according to our definitions we may say that the point P has *its abscissa equal to a, and its ordinate equal to b*; or, more briefly, *the co-ordinates of the point P are a and b*. We shall often speak of the point which has a for its abscissa and b for its ordinate, as *the point* (a, b).

3. A distance measured along the axis OX is however most frequently denoted by the symbol x, and a distance measured along the axis OY by the symbol y. Hence OX is called the axis of x, and OY the axis of y. Thus x and y are symbols to which we may ascribe different numerical values corresponding to the different points we consider, and we may express the statement that the co-ordinates of P are a and b, thus; for the point P, $x=a$ and $y=b$.

4. The lines $X'OX$, $Y'OY$, being indefinitely produced divide the plane in which they lie into four compartments. It becomes therefore necessary to distinguish points in one compartment from points in the others. For this purpose the following convention is adopted, which the reader has already seen in works on Trigonometry; lines measured along OX are considered *positive* and along OX' negative; lines measured along OY are considered positive, and along OY' negative. (See *Trigonometry*, Chap. IV.) If then we produce PN to a point Q such that $NQ=NP$, we have for the point Q, $x=-a$, $y=b$. If we produce PM to R so that $MR=MP$, we have for the point R, $x=a$, $y=-b$. Finally if we produce PO to S so that $OS=OP$, we have for the point S, $x=-a$, $y=-b$.

5. In the figure in Art. 1 we have taken the angle YOX a right angle; the axes are then called *rectangular*. If the angle YOX be not a right angle, the axes are called *oblique*. All that has been hitherto said applies whether the axes are rectangular or oblique. We shall always suppose the axes rectangular unless the contrary be stated; *this remark applies both to our investigations and to the examples which are given for the exercise of the student.*

6. Another method of determining the position of a point in a plane is by means of *polar co-ordinates.*



Let O be a fixed point, and OX a fixed line. Let P be any other point; join OP; then the position of P is determined if we know the angle XOP and the distance OP. The angle is usually denoted by θ and the distance by r.

O is called the *pole,* OX the *initial line;* OP the *radius vector* of the point P, and POX the *vectorial angle.*

7. The position of *any* point *might* be expressed by *positive* values of the polar co-ordinates θ and r, since there is here no ambiguity corresponding to that arising from the four compartments of the figure in Art. 4. It is however found convenient to use a similar convention to that in Art. 4; angles measured in one direction from OX are considered *positive* and in the other *negative.* Thus if in the figure XOP be a *positive* angle, XOQ will be a *negative* angle; if the angle XOQ be a quarter of a right angle, we may say that for XOQ, $\theta = -\frac{\pi}{8}$. It is, as we have stated, not absolutely *necessary* to introduce *negative* angles, but *convenient;* the position of the line OQ, for instance, might be determined by measuring from OX in the positive direction an angle $=2\pi - \frac{\pi}{8}$ as well as by measuring an angle in the negative direction $= \frac{\pi}{8}$.

Also positive and negative values of the radius vector are admitted. Thus, suppose the co-ordinates of P to be $\frac{\pi}{4}$ and a,

that is, let $XOP=\frac{\pi}{4}$ and $OP=a$; produce PO to P', so that $OP'=OP$, then P' *may* be determined by saying its co-ordinates are $\frac{\pi}{4}$ and $-a$. Thus when the radius vector is a negative quantity, we measure it on the same line as if it had been a positive quantity but in the *opposite* direction from O.

Hence if β represent any angle and c any length the *same* point is determined by the polar co-ordinates β and $-c$ as by the polar co-ordinates $\pi+\beta$ and c.

8. Let x, y denote the co-ordinates of P referred to OX as the axis of x, and a line through O perpendicular to OX as the axis of y. Also let θ and r be the polar co-ordinates of P. If we draw from P a perpendicular on OX, we see that

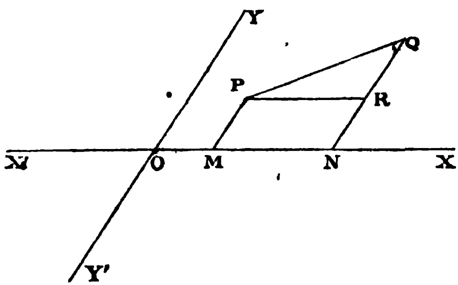
$$x=r\cos\theta,\quad \text{and}\quad y=r\sin\theta.$$

These equations connect the rectangular and polar co-ordinates of a point. From them, or from the figure, we may deduce

$$x^2+y^2=r^2,\quad \frac{y}{x}=\tan\theta.$$

9. We proceed to investigate expressions for some geometrical quantities in terms of co-ordinates.

To find an expression for the length of the line joining two points.



Let P and Q be the two points; ω the inclination of the axes OX, OY. Draw PM, QN parallel to OY; let x_1, y_1 be

the co-ordinates of P, and x_2, y_2, those of Q. Draw PR parallel to OX. Then, by Trigonometry,

$$PQ^2 = PR^2 + QR^2 - 2PR \,.\, QR \cos PRQ$$
$$= PR^2 + QR^2 + 2PR \,.\, QR \cos \omega.$$

But $PR = x_2 - x_1$, and $QR = y_2 - y_1$; therefore

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2 (x_2 - x_1)(y_2 - y_1) \cos \omega \ldots (1),$$

and thus the distance PQ is determined.

If the axes are rectangular, we have

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \ldots\ldots\ldots\ldots\ldots (2).$$

The student should draw figures placing P and Q in the different compartments and in different positions; the equations (1) and (2) will be found universally true.

From the equation (2) we have

$$PQ^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2 (x_1 x_2 + y_1 y_2) \ldots\ldots (3).$$

The following particular cases may be noted.

If P be at the origin $x_1 = 0$ and $y_1 = 0$; thus

$$PQ^2 = x_2^2 + y_2^2.$$

If P be on the axis of x and Q on the axis of y, $y_1 = 0$ and $x_2 = 0$; thus

$$PQ^2 = x_1^2 + y_2^2.$$

Let θ_1, r_1 be the polar co-ordinates of P, and θ_2, r_2 those of Q; then, by Art. 8,

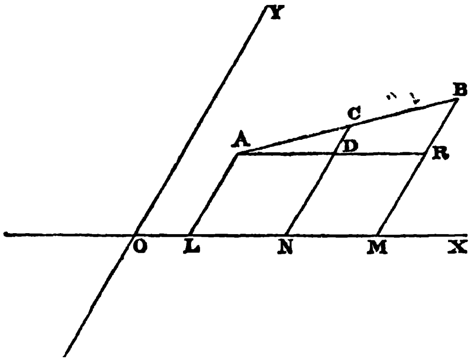
$$x_1 = r_1 \cos \theta_1, \qquad y_1 = r_1 \sin \theta_1,$$
$$x_2 = r_2 \cos \theta_2, \qquad y_2 = r_2 \sin \theta_2.$$

Substitute these values in (3) and we have

$$PQ^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos (\theta_2 - \theta_1).$$

This result can also be obtained immediately from the triangle POQ formed by drawing lines from P and Q to the origin.

10. *To find the co-ordinates of the point which divides in a given ratio the line joining two given points.*



Let A and B be the given points, x_1, y_1 the co-ordinates of A, and x_2, y_2 those of B; and let the required ratio be that of n_1 to n_2. Suppose C the required point, so that $AC : CB :: n_1 : n_2$. Draw the ordinates AL, BM, CN; and AR parallel to OX meeting CN in D. Let x, y be the co-ordinates of C.

It is obvious from the figure that

$$\frac{LN}{NM} = \frac{AD}{DR} = \frac{AC}{CB};$$

that is,

$$\frac{x - x_1}{x_2 - x} = \frac{n_1}{n_2};$$

$$\therefore x = \frac{n_1 x_2 + n_2 x_1}{n_1 + n_2}.$$

Similarly,

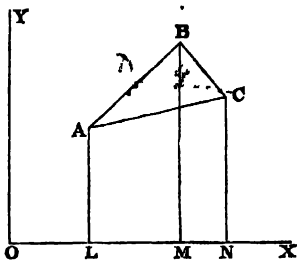
$$y = \frac{n_1 y_2 + n_2 y_1}{n_1 + n_2}.$$

In this article the axes may be oblique or rectangular. A simple case is that in which we require the co-ordinates of the point midway between two given points; then $n_1 = n_2$ and

$$x = \tfrac{1}{2}(x_1 + x_2), \qquad y = \tfrac{1}{2}(y_1 + y_2).$$

11. *To express the area of a triangle in terms of the co-ordinates of its angular points.*

Let ABC be a triangle; let x_1, y_1 be the co-ordinates of A; x_2, y_2 those of B; x_3, y_3 those of C. Draw the ordinates AL, BM, CN. The area of the triangle is equal to the trapezium $ABML$ + trapezium $BCNM$ − trapezium $ACNL$.



The area of the trapezium $ABML$ is $\frac{1}{2}LM(AL+BM)$. This is obvious, because if we join BL we divide the trapezium into two triangles, one having AL for its base and the other BM, and each having LM for its height;

thus, trapezium $ABML = \frac{1}{2}(x_2 - x_1)(y_1 + y_2)$;

also, trapezium $BCNM = \frac{1}{2}(x_3 - x_2)(y_2 + y_3)$;

and, trapezium $ACNL = \frac{1}{2}(x_3 - x_1)(y_1 + y_3)$;

therefore triangle ABC

$$= \frac{1}{2}\{(x_2 - x_1)(y_1 + y_2) + (x_3 - x_2)(y_2 + y_3) - (x_3 - x_1)(y_1 + y_3)\}.$$

This expression may be written more symmetrically thus;

$$\frac{1}{2}\{(x_2 - x_1)(y_2 + y_1) + (x_3 - x_2)(y_3 + y_2) + (x_1 - x_3)(y_1 + y_3)\} \ldots (1).$$

By reducing it, we shall find the area of the triangle

$$= \frac{1}{2}\{x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1\} \ldots\ldots\ldots (2).$$

If the axes be oblique and inclined at an angle ω, the area of the trapezium $ABML = \frac{1}{2}LM(AL+BM)\sin\omega$, and similarly for the other trapeziums. Thus the area of the triangle

will be found by multiplying the expressions given above by $\sin \omega$.

However the relative situations of A, B, C may be changed, the student will always find for the area of the triangle the expression (2), or that expression with the *sign of every term changed*. Hence we conclude, that we shall always obtain the area of the triangle by calculating the value of the expression (2), and changing the sign of the result if it should prove negative.

Locus of an equation. Equation to a curve.

12. Suppose an equation to be given between two unknown quantities, for example, $y - x - 2 = 0$. We see that this equation has an indefinite number of solutions, for we may assign to x any value we please, and from the equation determine the corresponding value of y. Thus corresponding to the values 1, 2, 3, ... of x, we have the values 3, 4, 5, ... of y. Now suppose a line, straight or curved, such that it passes through *every point* determined by giving to x and y values that satisfy the equation $y - x - 2 = 0$; such a line is called the *locus of the equation*. It will be shewn in the next chapter that the locus of the equation in question is a straight line. We shall see as we proceed that generally every equation between the quantities x and y has a corresponding locus.

But instead of starting with an equation and investigating what locus it represents, we may give a geometrical definition of a curve and deduce from that definition an appropriate equation; this will likewise appear as we proceed; we shall take successively different curves, define them, deduce their equations, and then investigate the properties of these curves by means of their equations. We shall in the next chapter begin with the *equation to a straight line.*

The connexion between a *locus* and an *equation* is the fundamental idea of the subject and must therefore be carefully considered; we shall place here a formal definition which we shall illustrate in the next chapter by applying it to a straight line.

DEF. The equation which expresses the invariable relation which exists between the co-ordinates of every point of

a curve is called the equation to the curve; and the curve, the co-ordinates of every point of which satisfy a given equation, is called the locus of that equation.

13. The student has probably already become familiar with the division of algebraical equations into equations of the first, second, third ... degree. When we speak of an equation of the n^{th} degree between two variables we mean that every term is of the form $Ax^{\alpha}y^{\beta}$ where α and β are zero or positive integers such that $\alpha+\beta$ is equal to n for one or more of the terms but not greater than n for any term, and A is a constant numerical quantity; and the equation is formed by connecting a series of such terms by the signs + and −, and putting the result = 0.

EXAMPLES. I

1. Find the polar co-ordinates of the points whose rectangular co-ordinates are

(1) $x=1,\ y=1$; (2) $x=-1,\ y=2$;

(3) $x=-1,\ y=1$; (4) $x=-1,\ y=-1$;

and indicate the points in a figure.

2. Find the rectangular co-ordinates of the points whose polar co-ordinates are

(1) $\theta=\frac{\pi}{3},\ r=3$; (2) $\theta=-\frac{\pi}{3},\ r=3$;

(3) $\theta=\frac{\pi}{3},\ r=-3$; (4) $\theta=-\frac{\pi}{3},\ r=-3$;

and indicate the points in a figure.

3. The co-ordinates of P are −1 and 4, and those of Q are 3 and 7; find the length of PQ.

4. Find the area of the triangle formed by joining the first three points in question 1.

5. A is a point on the axis of x and B a point on the axis of y; express the co-ordinates of the middle point of AB in terms of the abscissa of A and the ordinate of B; shew also that the distance of this point from the origin $= \frac{1}{2} AB$.

6. Transform equation (2) of Art. 11 so as to give an expression for the area of a triangle in terms of the *polar* co-ordinates of its angular points. Also obtain the result directly from the figure.

7. A and B are two points and O is the origin; express the area of the triangle AOB in terms of the co-ordinates of A and B, and also in terms of the polar co-ordinates of A and B.

8. A, B, C are three points the co-ordinates of which are expressed as in Art. 11; suppose D the middle point of AB; join CD and divide it in G so that $CG = 2GD$; find the co-ordinates of G.

9. Shew that each of the triangles GAB, GBC, GAC, formed by joining the point G in the preceding question to the points A, B, C, is equal in area to one-third of the triangle ABC. See Art. 11.

10. A and B are two points; the polar co-ordinates of A are θ_1, r_1; and those of B are θ_2, r_2. A line is drawn from the origin O bisecting the angle AOB; if C be the point where this line meets AB shew that the polar co-ordinates of C are $\theta = \frac{1}{2}(\theta_1 + \theta_2)$ and $r = \dfrac{2r_1r_2 \cos \frac{1}{2}(\theta_2 - \theta_1)}{r_1 + r_2}$.

11. Find the value of CD^2 and AD^2 in question 8 in terms of the co-ordinates there used; and shew that

$$AC^2 + BC^2 = 2CD^2 + 2AD^2.$$

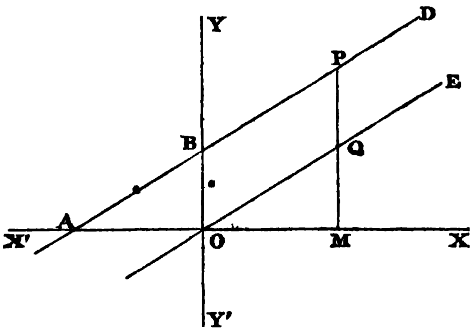
12. Find the value of GA^2, GB^2, and GC^2, in question 9 in terms of the co-ordinates there used; and shew that

$$3(GA^2 + GB^2 + GC^2) = AB^2 + BC^2 + CA^2.$$

CHAPTER II.

ON THE STRAIGHT LINE.

14. *To find the equation to a straight line.*



We shall first suppose the line not parallel to either axis.

Let ABD be a straight line meeting the axis of y in B. Draw a line OE through the origin parallel to ABD. In ABD take any point P; draw PM parallel to OY, meeting OX in M and OE in Q.

Suppose $OB = c$, and the tangent of $EOX = m$; and let x, y be the co-ordinates of P; then

$$\begin{aligned} y = PM &= PQ + QM \\ &= OB + QM \\ &= c + OM \tan QOM \\ &= c + mx. \end{aligned}$$

Hence the required equation is

$$y = mx + c.$$

OB is called the *intercept* on the axis of y; if the line

crosses the axis of y on the negative side of O, c will be negative.

We denote by m the tangent of the angle QOM or BAO, that is, the tangent of the angle which that part of the line which is above the axis of x makes with the axis of x produced in the positive direction. Hence if the line through the origin parallel to the given line falls between OY and OX, m is the tangent of an acute angle and is positive; if between OY and OX produced to the left, m is the tangent of an obtuse angle and is negative. So long as we consider the same straight line m and c remain unchangeable, they are therefore called *constant quantities* or *constants*. But x and y may have an *indefinite* number of values since we may ascribe to *one* of them, as x, any value we please, and find the *corresponding value* of y from the equation $y = mx + c$; x and y are therefore called *variable quantities* or *variables*.

If the line pass through the origin, $c = 0$, and the equation becomes

$$y = mx.$$

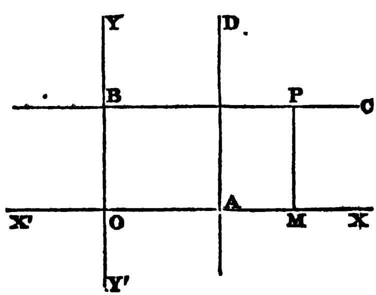
15. We have now to consider the cases in which the line is parallel to one of the axes.

If the line be parallel to the axis of x, $m = 0$, and the equation becomes

$$y = c.$$

If the line be parallel to the axis of y, m becomes the tangent of a right angle and is infinite; the preceding investigation is then no longer applicable. We shall now give separate investigations of these two cases.

To investigate the equation to a line parallel to one of the axes.



First suppose the line parallel to the axis of x. Let BC be the line meeting the axis of y in B; suppose $OB = b$.

Since the line is parallel to the axis of x, the ordinate PM of *any* point of it is equal to OB. Hence calling y the ordinate of any point P, we have for the equation to the line

$$y = b.$$

Next suppose the line parallel to the axis of y. Let AD be the line meeting the axis of x in A; suppose $OA = a$. Since the line is parallel to the axis of y, the abscissa of *any* point of it is OA. Hence calling x the abscissa of any point, we have for the equation to the line

$$x = a.$$

16. We have thus proved that any straight line whatsoever is represented by an equation of the first degree; we shall now shew that any equation of the first degree with two variables represents a straight line.

The general equation of the first degree with two variables is of the form

$$Ax + By + C = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

A, B, C being finite or zero.

First suppose B not zero; divide by B, then from (1)

$$y = -\frac{C}{B} - \frac{A}{B}x \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now we have seen in Art. 14, that if a line be drawn meeting the axis of y at a distance $-\frac{C}{B}$ from the origin and making with the axis of x an angle of which the tangent is $-\frac{A}{B}$, then (2) will be the equation to this line. Hence (2), and therefore also (1), represents a straight line.

If $A = 0$, then by Art. 15 the line represented by (1) is parallel to the axis of x.

If $B = 0$, then (1) becomes

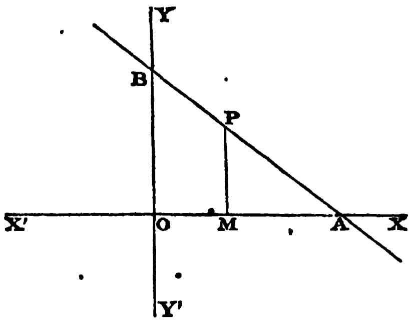
$$Ax + C = 0,$$

or
$$x = -\frac{C}{A},$$

and from Art. 15 we know that this equation represents a line parallel to the axis of y.

Hence the equation $Ax + By + C = 0$ always represents a straight line.

17. *Equation in terms of the intercepts.* The equation to a line may also be expressed in terms of its *intercepts* on the two axes.



Let A and B be the points where the straight line meets the axes of x and y respectively. Suppose $OA = a$, $OB = b$. Let P be any point in the line; x, y its co-ordinates; draw PM parallel to OY. Then by similar triangles,

$$\frac{PM}{OB} = \frac{AM}{AO};$$

that is,
$$\frac{y}{b} = \frac{a-x}{a};$$

$$\therefore \frac{x}{a} + \frac{y}{b} = 1.$$

18. It will be a useful exercise for the student to draw the straight lines corresponding to some given equations. Thus suppose the equation $2y + 3x = 7$ proposed; since a straight line is determined when two of its points are known, we may

find in any manner we please two points that lie on the line, and by joining them obtain the line. Suppose then $x=1$, it follows from the equation that $y=2$; hence the point which has its abscissa $=1$ and its ordinate $=2$ is on the line. Again, suppose $x=2$, then $y=\frac{1}{2}$; the point which has its abscissa $=2$ and its ordinate $=\frac{1}{2}$ is therefore on the line. Join the two points thus determined and the line so formed, produced indefinitely both ways, is the locus of the given equation. The two points that will be most easily determined are generally those in which the required line *cuts the axes.* Suppose $x=0$ in the given equation, then $y=\frac{7}{2}$, that is, the line passes through a point *on the axis of* y at a distance $\frac{7}{2}$ from the origin. Again, suppose $y=0$, then $x=\frac{7}{3}$, that is, the line passes through a point *on the axis of* x at a distance $\frac{7}{3}$ from the origin. Join the two points thus determined, and the line so formed, produced indefinitely both ways, is the locus of the given equation. What we have here ascertained as to the points where the line cuts the axis, may be obtained immediately from the equation; for if we write it in the form

$$\frac{3x}{7}+\frac{2y}{7}=1,$$

and compare it with the equation in Art. 17,

$$\frac{x}{a}+\frac{y}{b}=1,$$

we see that $a=\frac{7}{3}$ and $b=\frac{7}{2}$.

Again, suppose the equation $y=x$ proposed. Since this equation can be satisfied by supposing $x=0$ and $y=0$, the origin is a point of the line which the equation represents; therefore we need only determine *one other* point in it. Suppose $x=1$, then $y=1$; here another point is determined and the line can be drawn. The line may also be constructed by comparing the given equation with the form in Art. 14,

$$y=mx.$$

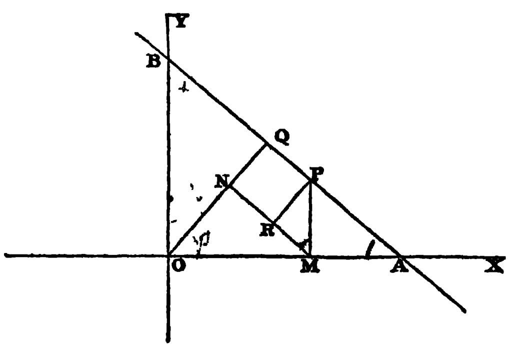
This we know represents a line passing through the origin and making with the axis of x an angle of which the tangent is m. Hence $y=x$ represents a line passing through the origin and inclined at an angle of 45° to the axis of x.

Similarly the equation $y = -x$ represents a line inclined to the axis of x at an angle of which the tangent is -1; that is, at an angle of 135°. Hence this equation represents a line through O bisecting the angle between OY and OX produced to the left in the figure to Art. 14.

19. The student is recommended to make himself thoroughly acquainted with the previous Articles before proceeding with the subject. In Algebra the theory of *indeterminate* equations does not usually attract much attention, and the student is sometimes perplexed on commencing a subject in which he has to consider *one* equation between two unknown quantities, which generally has an infinite number of solutions.

Our principal result up to the present point is, that a straight line corresponds to an equation of the first degree, and the student must accustom himself to perceive the appropriate line as soon as any equation is presented to him. The line can be determined by ascertaining two points through which it passes, that is, by finding two points such that the co-ordinates of each satisfy the given equation, and the line being thus determined, the co-ordinates of *any* point of it will satisfy the given equation.

20. *Equation to a straight line in terms of the perpendicular from the origin, and the inclination of this perpendicular to the axis.*



Let OQ be the perpendicular from the origin O on a line AB. Take any point P in the line; draw PM perpendicular to OA, MN perpendicular to OQ, and PR perpendicular to MN. Suppose $OQ = p$, and the angle $QOA = \alpha$. Let x, y be the co-ordinates of P; then

$$\begin{aligned} OQ = ON + NQ = ON + PR & \\ &= OM \cos QOA + PM \sin PMR \\ &= x \cos \alpha + y \sin \alpha. \end{aligned}$$

Therefore the equation to the line is

$$x \cos \alpha + y \sin \alpha = p.$$

21. We have given separate investigations of the different forms of the equation to a straight line in Articles 14, 17, 20; any one of these forms may however be readily deduced from either of the others by making use of the relations which exist between the constant quantities. The quantity which we have denoted by b in Art. 17, that is OB, is denoted by c in Art. 14;

$$\therefore b = c \text{..............................} (1).$$

In Art. 17,

$$\begin{aligned} \frac{b}{a} = \tan BAO = \tan (\pi - BAX) & \\ &= -\tan BAX; \end{aligned}$$

in Art. 14 we have denoted the tangent of BAX by m,

$$\therefore \frac{b}{a} = -m \text{....................} (2).$$

In Art. 20, $OA \cos \alpha = OQ$, and $OB \sin \alpha = OQ$; that is,

$$p = a \cos \alpha = b \sin \alpha \text{................} (3);$$

therefore from (2) and (3), $m = -\cot \alpha$.................(4).

Also if the equation

$$Ax + By + C = 0, \qquad -\frac{C}{B} = c$$

represent the straight line under consideration, then by Art. 16,

$$-\frac{A}{B}=m, \qquad -\frac{C}{B}=c=b \ldots\ldots\ldots\ldots\ldots(5);$$

$$\therefore \frac{A}{B}=\cot\alpha \text{ and } \frac{C}{B}=-\frac{p}{\sin\alpha} \ldots\ldots(6).$$

By means of these relations we may shew the agreement of the equations in Arts. 14, 17, 20, or from one of them deduce the others.

22. The student may exercise himself by varying the figures which we have used in investigating the equations. Thus, for example, in the figure to Art. 17, suppose the point P to be in BA produced, so that it falls *below* the axis of x. We shall still have

$$\frac{PM}{OB}=\frac{AM}{AO}, \quad \text{or } \frac{PM}{b}=\frac{x-a}{a}.$$

Now since P is below the axis of x, its ordinate y is a negative quantity, hence we must not put $PM=y$ but $PM=-y$, because by PM we mean a certain length estimated positively. Thus

$$-\frac{y}{b}=\frac{x-a}{a},$$

and therefore, as before,

$$\frac{x}{a}+\frac{y}{b}=1.$$

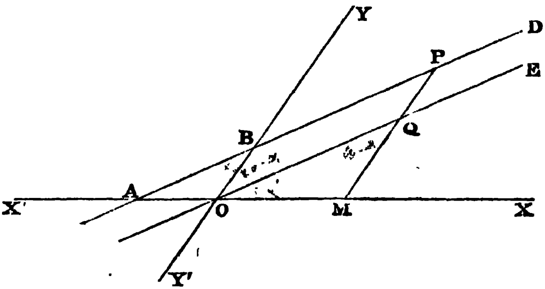
Oblique Co-ordinates.

23. *Equation to a straight line.*

We shall denote the inclination of the axes by ω.

Suppose first, that the line is not parallel to either axis. Let ABD be a straight line meeting the axis of y in B. Draw a line OE through the origin parallel to ABD. In ABD

take any point P; draw PM parallel to OY, meeting OX in M and OE in Q. Suppose $OB = c$, and the angle $QOM = \alpha$.



Let x, y be the co-ordinates of P; then

$$y = PM = PQ + QM = OB + QM.$$

But
$$\frac{QM}{OM} = \frac{\sin\alpha}{\sin(\omega - \alpha)};$$

$$\therefore\ QM = \frac{x \sin\alpha}{\sin(\omega - \alpha)}.$$

Hence the required equation is

$$y = \frac{x \sin\alpha}{\sin(\omega - \alpha)} + c.$$

If we put m for $\frac{\sin\alpha}{\sin(\omega - \alpha)}$ the equation becomes

$$y = mx + c,$$

as in Art. 14. The meaning of c is the same as before; m is the ratio of the sine of the inclination of the line to the axis of x to the sine of its inclination to the axis of y. Since $\sin\alpha$ is always positive, m will be positive or negative according as $\sin(\omega - \alpha)$ is positive or negative; thus as before m will be positive or negative according as the line through the origin parallel to the given line falls between OY and OX, or between OY and OX'. The meaning of m coincides with that in Art. 14 when $\omega = \frac{\pi}{2}$, for then $m = \tan\alpha$.

24. Since $$m = \frac{\sin \alpha}{\sin(\omega - \alpha)},$$

$$m(\sin\omega \cos\alpha - \cos\omega \sin\alpha) = \sin\alpha;$$

$$\therefore\ m(\sin\omega - \cos\omega \tan\alpha) = \tan\alpha;$$

$$\therefore\ \tan\alpha = \frac{m \sin\omega}{1 + m\cos\omega}.$$

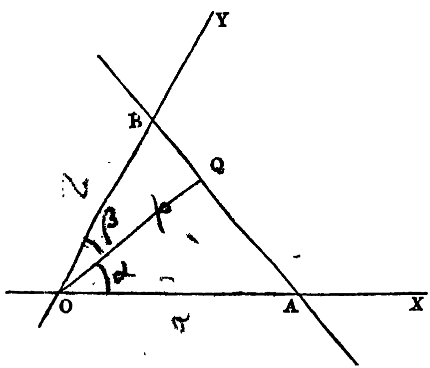
Hence $$\sin\alpha = \frac{m \sin\omega}{\pm\sqrt{(1 + 2m\cos\omega + m^2)}},$$

$$\cos\alpha = \frac{1 + m\cos\omega}{\pm\sqrt{(1 + 2m\cos\omega + m^2)}}.$$

Since $\sin\alpha$ is positive, we must take the upper or lower sign according as m is positive or negative.

25. The investigations in Arts. 15 and 17 apply without alteration to the case of oblique axes, and those in Art. 16 with the requisite change in the meaning of the constant m.

26. *To find the equation to a straight line in terms of the perpendicular from the origin, and the inclinations of the perpendicular to the axes.*



Let OQ be the perpendicular from the origin on a line AB; let $OQ = p$, $OA = a$, $OB = b$. If we suppose $QOA = \alpha$, we have $QOB = \omega - \alpha$; denote this by β; then

$$OQ = a \cos\alpha; \quad \therefore\ a = \frac{p}{\cos\alpha}.$$

$$OQ = b \cos\beta; \quad \therefore\ b = \frac{p}{\cos\beta}.$$

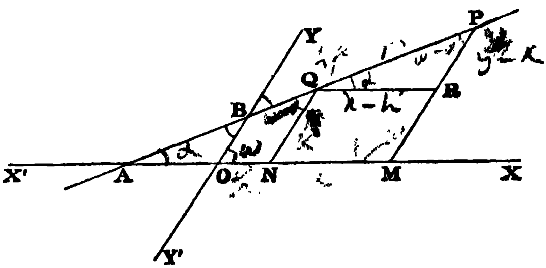
Substitute in the equation, Art. 17,

$$\frac{x}{a} + \frac{y}{b} = 1,$$

and we obtain

$$x \cos\alpha + y \cos\beta = p.$$

27. The following form of the equation to a straight line is often useful.



Let Q be a fixed point in any line AB; h, k its co-ordinates; let P be any other point in the line; x, y its co-ordinates; let $QP = r$, and the angle $BAX = \alpha$. Draw the ordinates PM, QN; and QR parallel to OX; then

$$\frac{x-h}{} = \frac{\sin(\omega - \alpha)}{\sin\omega} = l \text{ suppose,}$$

$$\frac{y-k}{r} = \frac{\sin\alpha}{\sin\cancel{\omega}(180-\omega)} = n \text{ suppose,}$$

thus $$\frac{x-h}{l} = \frac{y-k}{n}.$$

For the equation to the line it is sufficient to put

$$\frac{x-h}{l}=\frac{y-k}{n},$$

but it is useful to remember that each of these quantities is equal to r.

If the axes are rectangular, l and n become respectively $\cos\alpha$ and $\sin\alpha$, that is, the *cosines* of the inclinations of the line to the axes of x and y respectively.

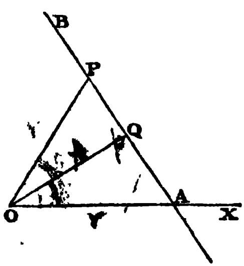
In the preceding figure P falls to the right of Q and $x-h$ is positive. If P were to the left of Q then $x-h$ would be negative. Thus since $x-h=lr$, the product lr must be capable of changing its sign; this leads us to consider r as positive or negative according to circumstances. When therefore we write the equation to a straight line under the form

$$\frac{x-h}{l}=\frac{y-k}{n},$$

and ascribe to l and n the values given above, we conclude that each of the expressions $\frac{x-h}{l}$ and $\frac{y-k}{n}$ is *numerically* equal to the distance between the point (h, k) and the point (x, y), but that the sign of each expression will depend upon the relative positions of the two points.

Polar Co-ordinates.

28. *Polar equation to a straight line.*



Let AB be a straight line, OQ the perpendicular on it from the origin, OX the initial line, P any point in the line.

Suppose $OQ = p$, and the angle $QOX = \alpha$. Let r, θ be the polar co-ordinates of P; then

$$OQ = OP \cos POQ;$$

that is, $$p = r\cos(\theta - \alpha).$$

This is the polar equation to the line.

29. The polar equation may also be derived from the equation referred to rectangular co-ordinates. Let

$$Ax + By + C = 0$$

be the equation to a line referred to rectangular co-ordinates. Put $r\cos\theta$ for x, and $r\sin\theta$ for y, Art. 8; thus

$$Ar\cos\theta + Br\sin\theta + C = 0 \quad \text{..............} \quad (1)$$

is the polar equation. This equation may be shewn to agree with

$$p = r\cos(\theta - \alpha) \quad \text{..................} \quad (2).$$

For by Art. 21 we have

$$\frac{A}{B} = \cot\alpha \text{ and } \frac{C}{B} = -\frac{p}{\sin\alpha}.$$

Hence (1) becomes

$$\cot\alpha\, r\cos\theta + r\sin\theta - \frac{p}{\sin\alpha} = 0,$$

which agrees with (2).

30. The equation to a line passing through the origin is, by Art. 14,

$$y = mx.$$

Put $r\cos\theta$ for x and $r\sin\theta$ for y; the equation then becomes

$$r\sin\theta = m\, r\cos\theta;$$

$$\therefore \tan\theta = m;$$

$$\therefore \theta = \text{a constant};$$

this is therefore the polar equation to a line passing through the origin.

31. We will collect here the different forms of the equation to a straight line which have been investigated,

$y = mx + c,$	Arts. 14 and 23.
$x = \text{constant, or, } y = \text{constant},$	Arts. 15 and 25.
$\frac{x}{a} + \frac{y}{b} - 1 = 0,$	Arts. 17 and 25.
$x \cos \alpha + y \sin \alpha - p = 0,$	Art. 20.
$y = \frac{\sin \alpha}{\sin(\omega - \alpha)} x + c,$	Art. 23.
$x \cos \alpha + y \cos \beta - p = 0,$	Art. 26.
$\frac{x-h}{l} = \frac{y-k}{n} = r,$	Art. 27.
$p = r \cos(\theta - \alpha),$	Art. 28.
$Ar \cos \theta + Br \sin \theta + C = 0,$	Art. 29.
$\theta = \text{constant},$	Art. 30.

EXAMPLES.

Draw the straight lines represented by the following equations:

(1) $y + 2x = 4;$ (2) $2y - x = 2;$

(3) $y + x = -2;$ (4) $x - 2y = 4;$

(5) $y + 2x = 0;$ (6) $1 = \cos\left(\theta - \frac{\pi}{4}\right);$

(7) $x = 1;$ (8) $\theta = \frac{\pi}{2};$

(9) $\theta = 0.$ (10) $\theta = 1.$

CHAPTER III.

PROBLEMS ON THE STRAIGHT LINE.

32. WE proceed to apply the results of the preceding articles to the solution of some problems.

To find the form of the equation to a straight line which passes through a given point.

Let x_1, y_1 be the co-ordinates of the given point, and suppose

$$y = mx + c \quad \text{.......................} (1)$$

to represent the straight line. Since the point (x_1, y_1) is on the line, its co-ordinates must satisfy (1); hence

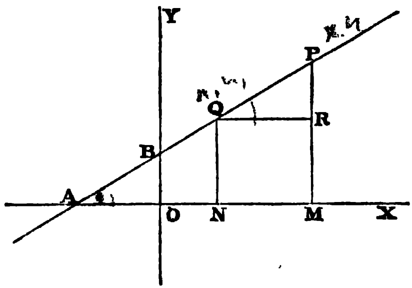
$$y_1 = mx_1 + c \quad \text{.......................} (2).$$

By subtraction,

$$y - y_1 = m(x - x_1) \quad \text{..................} (3);$$

this is the required equation.

33. The equation (3) of the preceding article obviously represents what is required, namely, a line passing through the point (x_1, y_1). For the equation is of the first degree in the variables x, y, and therefore, by Art. 16, must represent *some* straight line. Also the equation is obviously satisfied by the values $x = x_1$, $y = y_1$; that is, the line which the equation represents *does* pass through the given point. The constant m is the tangent of the angle which the line makes with the axis of x, and by giving a suitable value to m we may make the equation (3) represent *any* straight line which passes through the assigned point.



The geometrical meaning of equation (3) is obvious. For let AB be any straight line passing through the given point Q. Let P be any point in the line; x, y its co-ordinates. Draw the ordinates PM, QN; and QR parallel to OX; then

$$\frac{PR}{QR} = \text{tangent } PQR;$$

$$\text{that is, } \frac{y-y_1}{x-x_1} = \tan BAX = m,$$

which agrees with equation (3).

34. In Art. 32 we eliminated c between the equations (1) and (2) and retained m; we may if we please eliminate m and retain c. From (2)

$$m = \frac{y_1 - c}{x_1}.$$

Substitute in (1), thus

$$y = \frac{y_1 - c}{x_1} x + c;$$

$$\therefore\ yx_1 - xy_1 + c(x - x_1) = 0.$$

This equation obviously represents a straight line passing through the given point, because it is an equation of the first degree and is satisfied by the values $x = x_1$, $y = y_1$.

35. *To find the equation to the straight line which passes through two given points.*

Let x_1, y_1 be the co-ordinates of one given point; x_2, y_2 those of the other; suppose the equation to the straight line to be

$$y = mx + c \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since the line passes through (x_1, y_1) and (x_2, y_2),

$$y_1 = mx_1 + c \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$y_2 = mx_2 + c \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (2) by subtraction,

$$y - y_1 = m(x - x_1) \;\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

From (2) and (3) by subtraction,

$$y_2 - y_1 = m(x_2 - x_1),$$

$$\text{hence } m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Substitute the value of m in (4) and we have for the required equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \;\ldots\ldots\ldots\ldots\ldots (5).$$

We may also write the equation thus,

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1) \ldots\ldots\ldots (6).$$

Some particular cases may here be noted. Suppose $y_2 = y_1$, then (6) becomes $(x_2 - x_1)(y - y_1) = 0$, therefore $y = y_1$; the required line is thus parallel to the axis of x. Similarly if we suppose $x_2 = x_1$, then (6) becomes $(y_2 - y_1)(x - x_1) = 0$, therefore $x = x_1$; thus the required line is parallel to the axis of y. Lastly, suppose the point (x_1, y_1) to be the origin; hence $x_1 = 0$ and $y_1 = 0$; thus (6) becomes $x_2 y = y_2 x$. The student should illustrate these particular cases by figures.

36. The equation (6) of Art. 35 becomes by reduction

$$x_1 y - x y_1 + x_2 y_1 - x_1 y_2 + x y_2 - x_2 y = 0.$$

If we compare the expression on the left-hand side of this equation with the expression in brackets in equation (2) of

Art. 11, we see the only difference is that we have x and y in the place of x_3 and y_3 respectively. Thus the equation informs us that the area of the triangle formed by joining (x, y), (x_1, y_1), (x_2, y_2) vanishes, as should evidently be the case since the vertex (x, y) falls on the base, that is, on the line joining (x_1, y_1) to (x_2, y_2).

37. *To find the equation to the straight line which passes through a given point and divides the line joining two other given points in a given ratio.*

Let (h, k) be the first given point; let (x_1, y_1), (x_2, y_2) be the two other given points; let the given ratio in which the line joining the last two points is to be divided be that of n_1 to n_2; then, by Art. 10, the co-ordinates of the point of division are

$$\frac{n_1x_2 + n_2x_1}{n_1 + n_2}, \quad \frac{n_1y_2 + n_2y_1}{n_1 + n_2}.$$

Hence by equation (5) of Art. 35 the equation required is

$$y - k = \frac{\dfrac{n_1y_2 + n_2y}{n_1 + n_2} - k}{\dfrac{n_1x_2 + n_2x_1}{n_1 + n_2} - h}(x - h);$$

$$\text{or } y - k = \frac{n_1(y_2 - k) + n_2(y_1 - k)}{n_1(x_2 - h) + n_2(x_1 - h)}(x - h).$$

38. *To find the form of the equation to a straight line which is parallel to a given straight line.*

Let the equation to the given straight line be

$$y = m_1x + c_1 \quad\text{..................... (1)},$$

and the equation to the other straight line

$$y = mx + c \quad\text{........................(2)}.$$

Since the lines represented by (1) and (2) are parallel, they must have the same inclination to the axis of x; hence

$$m = m_1.$$

Thus (2) becomes

$$y=m_1x+c.$$

The quantity c remains undetermined since an indefinite number of straight lines can be drawn parallel to a given straight line.

39. *To determine the co-ordinates of the point of intersection of two given straight lines.*

Let the equation to one line be

$$y=m_1x+c_1 \qquad \text{.......................(1)},$$

and the equation to the other

$$y=m_2x+c_2 \qquad \text{.......................(2)}.$$

The co-ordinates of the point where the lines intersect must satisfy *both* equations; we must therefore find the values of x and y from (1) and (2). Thus

$$x=\frac{c_1-c_2}{m_2-m_1}, \quad y=\frac{c_1m_2-c_2m_1}{m_2-m_1};$$

these are the required co-ordinates.

40. *To find the condition in order that three straight lines may meet in a point.*

Let the equations to the lines be respectively

$$y=m_1x+c_1 \text{ ...... (1)}, \qquad y=m_2x+c_2 \text{ ...... (2)},$$
$$y=m_3x+c_3 \text{ ...... (3)}.$$

The co-ordinates of the point of intersection of (1) and (2) are

$$y=\frac{c_1m_2}{m}$$

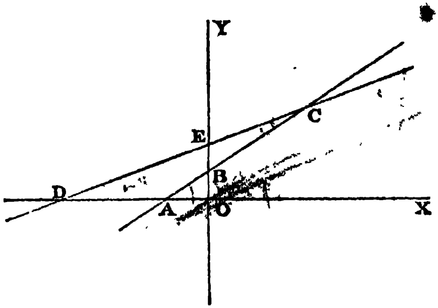
If the third line passes through the intersection of the first and second, these values must satisfy (3). Hence the necessary and sufficient condition is

$$\frac{c_1m_2-c_2m_1}{m_2-m_1}=\frac{m_3(c_1-c_2)}{m_2-m_1}+c_3,$$

that is,

$$c_1m_2 - c_2m_1 + c_2m_3 - c_3m_2 + c_3m_1 - c_1m_3 = 0.$$

41. *To find the angle between two given straight lines.*



Let ABC be one line and DEC the other; let the equation to the former be

$$y = m_1x + c_1,$$

and the equation to the latter

$$y = m_2x + c_2.$$

Then $$\tan ACD = \tan(CAX - CDX)$$

$$= \frac{\tan CAX - \tan CDX}{1 + \tan CAX \tan CDX}$$

$$= \frac{m_1 - m_2}{1 + m_1m_2}.$$

From this we may deduce

$$\cos ACD = \frac{1 + m_1m_2}{\sqrt{\{(1 + m_1^2)(1 + m_2^2)\}}},$$

$$\sin ACD = \frac{m_1 - m_2}{\sqrt{\{(1 + m_1^2)(1 + m_2^2)\}}}.$$

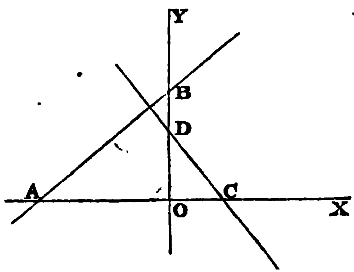
42. *To find the form of the equation to a straight line which is perpendicular to a given straight line.*

Let $$y = mx + c$$
be the equation to the given line, and
$$y = m'x + c'$$
the equation to another line. Then the tangent of the angle between these lines is
$$\frac{m - m'}{1 + mm'}.$$

If these lines are perpendicular,
$$1 + mm' = 0;$$
$$\therefore\ m' = -\frac{1}{m}.$$

Hence $$y = -\frac{x}{m} + c'$$
represents a line perpendicular to the line
$$y = mx + c.$$

43. The result of the last article may also be obtained thus,



Let AB be the given line, so that $\tan BAX = m$. Let CD be a line perpendicular to AB; then
$$\tan DCX = -\tan DCO$$
$$= -\cot BAO$$
$$= -\frac{1}{m}.$$

Hence the equation to CD is

$$y = -\frac{x}{m} + c',$$

where $$c' = OD.$$

44. *To find the equation to the straight line which passes through a given point, and is perpendicular to a given straight line.*

Let x_1, y_1 be the co-ordinates of the given point, and

$$y = mx + c \quad \text{.......................} \quad (1)$$

the equation to the given line. The form of the equation to a line through (x_1, y_1) is

$$y - y_1 = m'(x - x_1) \quad \text{..................} \quad (2).$$

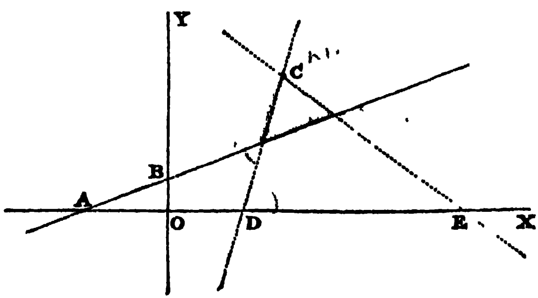
If (2) is perpendicular to (1), we have

$$m'm + 1 = 0.$$

Hence the required equation is

$$y - y_1 = -\frac{1}{m}(x - x_1).$$

45. *To find the equations to the straight lines which pass through a given point and make a given angle with a given straight line.*



Let AB be the given straight line; C the given point; h, k its co-ordinates; β the given angle. Let the equation to AB be

$$y = mx + c.$$

Suppose CD and CE the two lines which can be drawn through C, each making an angle β with AB. Then

$$\tan CDX = \tan (BAX + \beta)$$

$$= \frac{m + \tan\beta}{1 - m\tan\beta},$$

$$\tan CEX = -\tan CEA = -\tan (\beta - BAX)$$

$$= \frac{m - \tan\beta}{1 + m\tan\beta}.$$

Hence the equation to CD is

$$y - k = \frac{m + \tan\beta}{1 - m\tan\beta}(x - h);$$

and the equation to CE is

$$y - k = \frac{m - \tan\beta}{1 + m\tan\beta}(x - h).$$

46. The following particular cases of the preceding results may be noted.

(1) Suppose $m = 0$; then the given line is parallel to the axis of x. The required equations then are

$$y - k = \tan\beta\,(x - h),$$

$$\text{and } y - k = -\tan\beta\,(x - h).$$

(2) Suppose $m = \infty$; then the given line is parallel to the axis of y. And since

$$\frac{m + \tan\beta}{1 - m\tan\beta} = \frac{1 + \frac{1}{m}\tan\beta}{\frac{1}{m} - \tan\beta},$$

we have when $m = \infty$, and therefore $\frac{1}{m} = 0$, for the equation to CD,

$$y - k = -\frac{1}{\tan\beta}(x - h) = -\cot\beta\,(x - h).$$

Similarly the equation to CE becomes

$$y - k = \cot\beta\,(x - h).$$

(3) Suppose $m = \tan\beta$. In this case the equation to CD becomes

$$y - k = \frac{2\tan\beta}{1 - \tan^2\beta}(x - h),$$

that is, $$y - k = \tan 2\beta\,(x - h).$$

The equation to CE becomes

$$y - k = 0,$$

so that CE is parallel to the axis of x.

(4) Suppose $m = \cot\beta$. The equation to CD may be written in the form

$$(y - k)(1 - m\tan\beta) = (m + \tan\beta)(x - h),$$

and we see that when $m = \cot\beta$ the left-hand side is zero; thus the required equation is then

$$x - h = 0.$$

The equation to CE becomes

$$y - k = \frac{\cot\beta - \tan\beta}{2}(x - h)$$

$$= \frac{\cos^2\beta - \sin^2\beta}{2\cos\beta\sin\beta}(x - h)$$

$$= \cot 2\beta\,(x - h).$$

(5) Suppose $m = -\tan\beta$. Then the equation to CD becomes

$$y - k = 0,$$

and the equation to CE becomes

$$y-k=\frac{-2\tan\beta}{1-\tan^2\beta}(x-h)$$

$$=-\tan 2\beta\,(x-h).$$

(6) Suppose $m=-\cot\beta$. Then the equation to CD becomes

$$y-k=\frac{\tan\beta-\cot\beta}{2}(x-h)$$

$$=-\cot 2\beta\,(x-h).$$

The equation to CE may be written in the form

$$(y-k)(1+m\tan\beta)=(m-\tan\beta)(x-h),$$

and we see that when $m=-\cot\beta$ the left-hand member is zero; thus the required equation is then

$$x-h=0.$$

(7) Suppose $\beta=\frac{\pi}{2}$. The equation to CD may be written

$$y-k=\frac{m\cot\beta+1}{\cot\beta-m}(x-h).$$

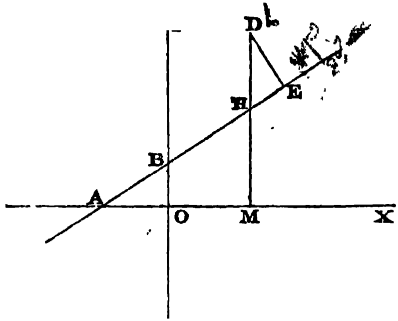
When $\beta=\frac{\pi}{2}$ we have $\cot\beta=0$; thus the equation becomes

$$y-k=-\frac{1}{m}(x-h).$$

Similarly the equation to CE takes the same form; and thus the result agrees with that of Art. 44.

We have discussed these particular cases as an example of the manner in which the student should test his comprehension of the subject by applying the general formulæ to special examples. He will find it useful to illustrate these cases by figures.

47. *To find the length of the perpendicular drawn from a given point upon a given straight line.*



Let AB be the given straight line; D the given point; h, k its co-ordinates. Let the equation to AB be

$$y = mx + c \text{ .... } (1).$$

The equation to the line through D perpendicular to AB is, by Art. 44,

$$y - k = -\frac{1}{m}(x - h) \text{ .... } (2).$$

Let x_1, y_1 be the co-ordinates of E; then, by Art. 9,

$$DE^2 = (x_1 - h)^2 + (y_1 - k)^2 \text{ .... } (3).$$

It remains then to substitute for x_1 and y_1 their values in (3). Now, since x_1, y_1 are the co-ordinates of E, which is the point where (1) and (2) meet, we have

$$y_1 = mx_1 + c, \text{ and } y_1 - k = -\frac{1}{m}(x_1 - h);$$

$$\therefore mx_1 + c = k - \frac{1}{m}(x_1 - h);$$

$$\therefore x_1 = \frac{mk + h - mc}{1 + m^2},$$

$$\text{and } x_1 - h = \frac{mk - m^2h - mc}{1 + m^2}$$

$$\frac{m}{1+m^2}(k - mh - c).$$

Also $$y_1 = mx_1 + c = \frac{m^2k + mh + c}{1 + m^2}$$

and $$y_1 - k = \frac{mh + c - k}{1 + m^2};$$

$\therefore$ by (3) $$DE^2 = \frac{m^2}{(1+m^2)^2}(k - mh - c)^2 + \frac{(k - mh - c)^2}{(1+m^2)^2}$$

$$\frac{(k - mh - c)^2}{1 + m^2}.$$

Hence $$DE = \frac{k - mh - c}{\sqrt{(1+m^2)}}.$$

The radical in the denominator may be taken with the positive or negative sign, according as the numerator is positive or negative, so as to give for DE a positive value.

We may also obtain the value of DE thus; draw the ordinate DM meeting the line AB in H; then

$$DE = DH \sin DHE = DH \cos HAM.$$

Now $OM = h$; $\therefore HM = mh + c$, and $DM = k$;

$$\therefore DH = k - mh - c.$$

Also $\tan HAM = m$; $\quad \therefore \cos HAM = \dfrac{1}{\sqrt{(1+m^2)}}$;

$$\therefore DE = \frac{k - mh - c}{\sqrt{(1+m^2)}}.$$

Hence if on the line $y - mx - c = 0$ a perpendicular be drawn from the point (h_1, k_1) and also a perpendicular from the point (h_2, k_2), the ratio of the first perpendicular to the second is equal to the numerical ratio of $k_1 - mh_1 - c$ to $k_2 - mh_2 - c$.

48. *To find the length of the line drawn from a given point in a given direction to meet a given line.*

Let (h, k) be the given point; and suppose a line drawn from this point at an inclination α to the axis of x to meet the line

$$Ax + By + C = 0 \quad\text{.......................}(1).$$

Let r be the required length; x_1, y_1 the co-ordinates of the point where the line drawn from (h, k) meets (1); then, by Art. 27,

$$x_1 - h = r\cos\alpha, \qquad y_1 - k = r\sin\alpha \quad\text{.........}(2).$$

But (x_1, y_1) is on (1),

$$\therefore\ A(h + r\cos\alpha) + B(k + r\sin\alpha) + C = 0;$$

$$r = -\frac{Ah + Bk + C}{A\cos\alpha + B\sin\alpha}.$$

49. In this chapter we have used equations of the form $y = mx + c$ to represent straight lines. The student may exercise himself by solving the problems by means of the more symmetrical forms of the equation to a straight line,

$$Ax + By + C = 0,$$

$$\frac{x}{a} + \frac{y}{b} - 1 = 0,$$

$$x\cos\alpha + y\sin\alpha - p = 0.$$

The results of course can be easily compared with those we have obtained. For example, if in Art. 47 we represent the given line by the equation

$$Ax + By + C = 0,$$

the result obtained should coincide with the value of

$$\frac{k - mh - c}{\sqrt{(1 + m^2)}}$$

when for m we write $-\dfrac{A}{B}$ and $-\dfrac{C}{B}$ for c; that is, the result must be

$$\frac{Ah + Bk + C}{\sqrt{(A^2 + B^2)}}.$$

Similarly, if the given line be represented by

$$x \cos \alpha + y \sin \alpha - p = 0,$$

we shall find for the perpendicular on it from (h, k)

$$\pm (h \cos \alpha + k \sin \alpha - p).$$

Thus if the equation to a line be in the form

$$x \cos \alpha + y \sin \alpha - p = 0,$$

the length of the perpendicular drawn from a point on this line is the numerical *value of the expression on the left-hand side of this equation, when for* x *and* y *are substituted the co-ordinates of the given point.* This result is of such great importance that we shall proceed to examine it more closely.

50. We may however previously observe that if the equation to a line be given in any form, we can immediately transform it so that it may be expressed in terms of the length of the perpendicular from the origin and the inclination of this perpendicular to the axis of x. For example, suppose the equation to be

$$2x + 3y + 4 = 0.$$

Change the sign of every term so that the last term may be negative; thus the equation becomes

$$-2x - 3y - 4 = 0.$$

Divide by $\sqrt{(2^2 + 3^2)}$; thus

$$-\frac{2x}{\sqrt{13}} - \frac{3y}{\sqrt{13}} - \frac{4}{\sqrt{13}} = 0.$$

This is of the form

$$x \cos \alpha + y \sin \alpha - p = 0,$$

$$\text{and } \cos \alpha = -\frac{2}{\sqrt{13}}, \quad \sin \alpha = -\frac{3}{\sqrt{13}}, \quad p = \frac{4}{\sqrt{13}}.$$

In this example α is an angle lying between π and $\frac{3\pi}{2}$.

Any other example may be treated in a similar manner—the rule being the following. Collect the terms on one side, and if necessary, change the signs so that the equation may be in the form $Ax+By-C=0$, where C is positive; then divide by $\sqrt{(A^2+B^2)}$; thus the equation becomes

$$\frac{Ax}{\sqrt{(A^2+B^2)}}+\frac{By}{\sqrt{(A^2+B^2)}}-\frac{C}{\sqrt{(A^2+B^2)}}=0;$$

this is of the required form, and

$$\cos\alpha=\frac{A}{\sqrt{(A^2+B^2)}},\quad \sin\alpha=\frac{B}{\sqrt{(A^2+B^2)}},\quad p=\frac{C}{\sqrt{(A^2+B^2)}}.$$

Thus every equation representing a straight line may be brought to the form

$$x\cos\alpha+y\sin\alpha-p=0,$$

where p is a *positive* quantity, unless the line passes through the origin, and then $p=0$.

When we use the equation

$$x\cos\alpha+y\sin\alpha-p=0$$

we shall always suppose p *positive.*

51. The line whose equation is

$$x\cos\alpha+y\sin\alpha-p=0$$

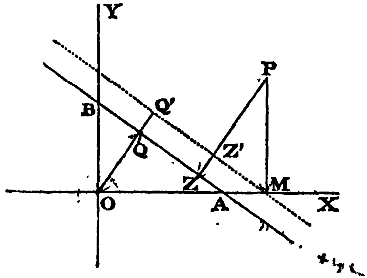
might be called "*the line* (p, α)," since the constants p and α determine the line; but when there is no risk of confounding it with another line, it may be more shortly called "*the line* α," and the equation may be expressed shortly, thus, "$\alpha=0$."

We shall now give another investigation of the expression for the perpendicular from a given point on the line (p, α).

Let AB represent the line (p, α), O the origin, P the point (x, y), so that P and O are on *opposite* sides of AB. Draw OQ, PZ perpendicular to AB, and PM parallel to OY; through M draw a line parallel to AB, meeting OQ and PZ, produced if necessary, in Q' and Z' respectively.

Then $OQ' = OM \cos\alpha = x\cos\alpha$; $PZ' = PM \sin\alpha = y\sin\alpha$;

$$PZ = OQ' + PZ' - OQ = x\cos\alpha + y\sin\alpha - p.$$



If P and O be on the *same* side of AB we shall obtain for the perpendicular

$$p - x\cos\alpha - y\sin\alpha.$$

It will be found that these results will hold for all varieties of the figure.

52. Or we may proceed as follows.

Let $$x\cos\alpha + y\sin\alpha - p = 0 \text{.......................} (1)$$

be the equation to a straight line, and let x', y' be the co-ordinates of the point from which a perpendicular is drawn upon the line; it is required to find the length of this perpendicular. The equation to any line which is parallel to (1) and on the same side of the origin, may be written thus,

$$x\cos\alpha + y\sin\alpha - p' = 0 \text{..................} (2)$$

where p' is the perpendicular from the origin upon this line. If this line pass through the point (x', y'), we must have

$$x'\cos\alpha + y'\sin\alpha - p' = 0;$$

$$\therefore\ p' = x'\cos\alpha + y'\sin\alpha.$$

The length of the perpendicular from (x', y') on (1) wil[l be] $p' - p$ if the point and the origin are on different sides of [the] line, and $p - p'$ if they are on the same side; that is,

$$x'\cos\alpha + y'\sin\alpha - p$$

in the former case, and in the latter case

$$p - x'\cos\alpha - y'\sin\alpha.$$

If the line parallel to (1) be on the opposite side of the origin, its equation will be

$$x \cos(\pi + \alpha) + y \sin(\pi + \alpha) - p' = 0,$$

where p' is the length of the perpendicular from the origin upon it. If this line pass through the point (x', y') we must have

$$x' \cos \alpha + y' \sin \alpha + p' = 0;$$

$$\therefore p' = -x' \cos \alpha - y' \sin \alpha.$$

The length of the perpendicular from (x', y') on (1) will be the sum of p' and p, that is,

$$p - x' \cos \alpha - y' \sin \alpha.$$

We may now suppress the accents on x and y, and we have the same conclusion as before.

53. Thus the perpendicular from the point (x, y) on the line

$$x \cos \alpha + y \sin \alpha - p = 0$$

is $\quad x \cos \alpha + y \sin \alpha - p, \quad$ or $\quad p - x \cos \alpha - y \sin \alpha,$

according as the point (x, y) and the origin are on different sides of the line or on the same side of it.

The student will perceive that we speak here of the point (x, y) and the line $x \cos \alpha + y \sin \alpha - p = 0$, and that we use the same *symbols* x, y, in speaking of the point and of the line. But we do not mean that the point (x, y) is to be *on* the line, that is, we do not mean the x and y which are co-ordinates of the point (x, y) to have the *same values* as they have for any point in the line $x \cos \alpha + y \sin \alpha - p = 0$. We might just use x', y' as co-ordinates of the point to prevent con[illegible]n, but it is found convenient to adopt the notation here as the advantages more than compensate for the increased attention which is required from the student in distinguishing the different meanings of the symbols.

54. We have in Art. 51 left the student to convince himself by drawing the figures in different ways, that the perpendicular from the point (x, y) on the line (p, α) is *always*

$$\pm (x \cos \alpha + y \sin \alpha - p),$$

the upper or lower sign being taken according as (x, y) and the origin are on *different* sides, or on the *same* side of the line (p, α). We may also arrive at the result imperfectly, thus. We may first prove, as in Art. 47, that the perpendicular must always be equal to one of the two expressions

$$\pm (x \cos \alpha + y \sin \alpha - p),$$

and may then proceed to distinguish the cases. Now the expression $x \cos \alpha + y \sin \alpha - p$ is *negative* when the point (x, y) is the origin, because it becomes then $-p$; also this expression cannot change its sign so long as (x, y) is taken on the same side of the line (p, α) as the origin *because it cannot change its sign without passing through the value zero,* and it cannot vanish until the point (x, y) is on the line. Hence the expression remains negative so long as (x, y) is on the same side of the line (p, α) as the origin. Similarly, if the expression is positive when the point (x, y) has *any one* position on the *other* side of the line (p, α), it will continue positive so long as (x, y) is on that side of the line; and it may be easily shewn that the expression *can* be made positive by suitable values of x and y; hence it *is* always positive while (x, y) is on the opposite side from the origin. We call this an imperfect method, because the sentence in italics on which the method depends, has probably not sufficiently attracted the student's attention up to this period of his studies to produce perfect conviction.

55. If the equation to a line be $x \cos \alpha + y \sin \alpha = 0$, so that $p = 0$, we shall still have $\pm (x \cos \alpha + y \sin \alpha)$ as the length of the perpendicular from the point (x, y) on it. We may discriminate as follows, let the equation be so written that the coefficient of y is *positive;* then for points on the same side of the line as the *positive* part of the axis of y, the perpendicular is $x \cos \alpha + y \sin \alpha$; for points on the other side it is $-(x \cos \alpha + y \sin \alpha)$. This is easily shewn by comparing a few figures, or as in Art. 54.

Oblique Axes.

56. The results in Arts. 32—40 hold whether the axes are rectangular or oblique; in Art. 33, however, m must have that meaning which is required when the axes are oblique.

To find the angle between two straight lines referred to oblique axes.

Let ω be the angle between the axes; $y = m_1 x + c_1$ the equation to one line; $y = m_2 x + c_2$ the equation to the other. Let α_1, α_2 be the angles which these lines make with the axis of x; and β the angle between them.

By Art. 24

$$\tan \alpha_1 = \frac{m_1 \sin \omega}{1 + m_1 \cos \omega}; \quad \tan \alpha_2 = \frac{m_2 \sin \omega}{1 + m_2 \cos \omega}.$$

Hence $\tan \beta$ or $\tan (\alpha_2 - \alpha_1) = \dfrac{\dfrac{m_2 \sin \omega}{1 + m_2 \cos \omega} - \dfrac{m_1 \sin \omega}{1 + m_1 \cos \omega}}{1 + \dfrac{m_1 m_2 \sin^2 \omega}{(1 + m_1 \cos \omega)(1 + m_2 \cos \omega)}}$

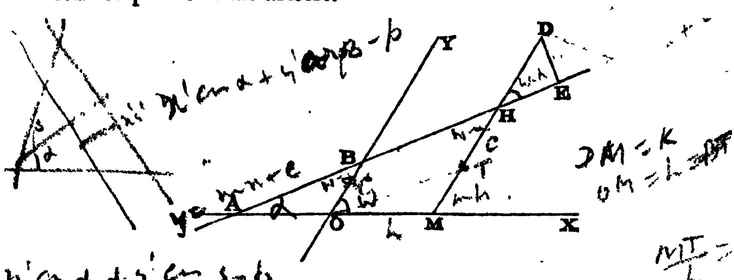
$$= \frac{(m_2 - m_1) \sin \omega}{1 + (m_1 + m_2) \cos \omega + m_1 m_2}.$$

Hence the condition that the lines may be at right angles is

$$1 + (m_1 + m_2) \cos \omega + m_1 m_2 = 0.$$

57. *To find the length of the perpendicular drawn from a given point on a given straight line.*

We shall proceed as in the latter part of Art. 47; the student may also obtain the result by the method in the former part of that article.



Let AB be the given straight line; D the given point; h, k its co-ordinates.

Let the equation to AB be

$$y = mx + c.$$

Draw DHM parallel to OY, and DE perpendicular to AB; then

$$DE = DH \sin DHE.$$

Now $\quad DH = DM - HM = k - (mh + c) = k - mh - c.$

Let $\quad BAX = \alpha, \quad$ then DHE or $AHM = \omega - \alpha$,

$$\text{and } \frac{\sin \alpha}{\sin(\omega - \alpha)} = m \quad \text{(Art. 24)};$$

$$\therefore \sin(\omega - \alpha) = \frac{\sin \alpha}{m} = \frac{\sin \omega}{\sqrt{(1 + 2m \cos \omega + m^2)}} \text{ (Art. 24)};$$

$$\therefore DE = \frac{(k - mh - c) \sin \omega}{\sqrt{(1 + 2m \cos \omega + m^2)}}.$$

If a line be drawn from D to meet AB at an angle β, its length will be $DE \operatorname{cosec} \beta$, and will therefore be known since DE is known.

If the equation to a straight line be in the form given in Art. 26, namely,

$$x \cos \alpha + y \cos \beta - p = 0,$$

the length of the perpendicular on it from the point (x', y') will be

$$\pm (x' \cos \alpha + y' \cos \beta - p).$$

This may be deduced from the preceding expression, or it may be obtained in the manner of Art. 51.

Polar Co-ordinates.

58. *To find the polar equation to the straight line which passes through two given points.*

Let r_1, θ_1 be the co-ordinates of one point; and r_2, θ_2 those of the other; and suppose the equation to the line

$$r \cos(\theta - \alpha) = p,$$

that is, $$r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = p \quad\text{.................(1).}$$

Since this line passes through the two points, we have

$$r_1\cos\theta_1\cos\alpha + r_1\sin\theta_1\sin\alpha = p \quad\text{...............(2),}$$

$$r_2\cos\theta_2\cos\alpha + r_2\sin\theta_2\sin\alpha = p \quad\text{...............(3).}$$

From (1) and (2)

$$(r\cos\theta - r_1\cos\theta_1)\cos\alpha + (r\sin\theta - r_1\sin\theta_1)\sin\alpha = 0 \quad\text{...(4).}$$

From (2) and (3)

$$(r_1\cos\theta_1 - r_2\cos\theta_2)\cos\alpha + (r_1\sin\theta_1 - r_2\sin\theta_2)\sin\alpha = 0 \quad\text{...(5),}$$

$$\therefore \frac{r\cos\theta - r_1\cos\theta_1}{r_1\cos\theta_1 - r_2\cos\theta_2} = \frac{r\sin\theta - r_1\sin\theta_1}{r_1\sin\theta_1 - r_2\sin\theta_2}.$$

After reduction we obtain

$$rr_1\sin(\theta_1 - \theta) + r_1r_2\sin(\theta_2 - \theta_1) + r_2r\sin(\theta - \theta_2) = 0 \quad\text{...(6).}$$

This equation has a simple geometrical interpretation; for if we draw a figure and take O for the origin, and A, B, P for the points (r_1, θ_1), (r_2, θ_2), (r, θ), respectively, we see that equation (6) is the expression of the fact that one of the triangles OAP, OBP, OAB, is equal in area to the sum of the other two.

59. We have seen that a straight line is the locus of an equation of the first degree; as we proceed it will appear that if an equation be of a degree higher than the first, the corresponding locus will be *generally* some curve; we may notice here some exceptional cases.

Suppose the equation

$$x^2 - 4ax + 4a^2 + y^2 = 0$$

be proposed; this equation may be written

$$(x - 2a)^2 + y^2 = 0.$$

Hence we see that the *only* solution is

$$y = 0,\ x = 2a.$$

Thus the corresponding locus consists only of a single point on the axis of x at a distance $2a$ from the origin.

Again, suppose the equation to be

$$x^2+y^2+1=0.$$

No real values of x and y will satisfy this equation; in this case then there is no corresponding locus, or as it is usually expressed, the locus is *impossible*. Thus, the locus corresponding to a given equation *may* reduce to a single point, or it may be impossible.

60. We have seen that the equation to a single straight line is always of the *first* degree; an equation of a higher degree than the first *may* however represent a locus consisting of two or more straight lines. For example, suppose

$$y^2-x^2=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1);$$

$$\therefore\; y=x \;\ldots\ldots\ldots\; (2), \quad \text{or } y=-x \;\ldots\ldots\ldots\ldots\; (3).$$

If the co-ordinates of a point satisfy *either* (2) *or* (3), they will satisfy (1); that is, every point which is comprised in the locus (2) is comprised in (1), and every point which is comprised in (3) is also comprised in (1). Hence (1) represents *two* straight lines which pass through the origin, and make respectively angles of 45° and 135° with the axis of x.

61. An equation which only involves *one* of the variables, represents a series of lines parallel to one of the axes. Thus, if there be an equation $f(x)=0$, we obtain by solving it a series of values for x, as $x=a_1$ or $x=a_2$,...... and each of these equations represents a line parallel to the axis of y. Similarly $f(y)=0$ represents a series of lines parallel to the axis of x.

An equation of the form $f\left(\frac{y}{x}\right)=0$ represents a series of lines passing through the origin; for by solving the equation we obtain a series of values for $\frac{y}{x}$, as $\frac{y}{x}=m_1$, $\frac{y}{x}=m_2$,.... and each of these equations represents a line passing through the

origin. Of course if an equation $f(x)=0$, $f(y)=0$, or $f\left(\frac{y}{x}\right)=0$ have no real roots, the corresponding locus is impossible.

The equation

$$Ay^2+Bxy+Cx^2=0$$

may be put in the form

$$A\left(\frac{y}{x}\right)^2+B\frac{y}{x}+C=0.$$

Since this is a quadratic in $\frac{y}{x}$ we obtain *two* values for it, suppose $\frac{y}{x}=m_1$ and $\frac{y}{x}=m_2$; hence the equation generally represents *two* straight lines passing through the origin. If B^2 be less than $4AC$, then m_1 and m_2 are impossible, and the *only* solution of the given equation is $x=0$, $y=0$; that is, the locus is a single point, namely, the origin.

62. It is obvious that if the locus represented by an equation $f(x, y)=0$ passes through the origin, the values $x=0$, $y=0$ must satisfy the equation. We can thus immediately determine by inspection whether a proposed locus passes through the origin or not.

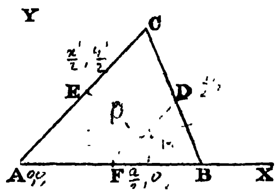
63. In Art. 39 we determined the co-ordinates of the point of intersection of two given straight lines: the proposition may obviously be generalised. Let $f_1(x, y)=0, f_2(x, y)=0$, represent two curves, then the co-ordinates of the points where they meet will be determined by solving these simultaneous equations. It may be shewn that if one equation be of the m^{th} degree and the other of the n^{th}, the number of common points cannot exceed mn. (See *Theory of Equations*, Chapter XX.)

64. We will exemplify the articles of this chapter by applying them to prove some properties of a triangle.

The lines drawn from the angles of a triangle to the middle points of the opposite sides meet in a point.

Let ABC be a triangle, D, E, F the middle points of the sides; take A for the origin, AB for the direction of the axis

of x, and a line through A perpendicular to AB for the axis of y. Let $AB = a$, and let x', y' be the co-ordinates of C.



Since D is the middle point of CB, the abscissa of D is $\frac{1}{2}(x' + a)$ and its ordinate $\frac{y'}{2}$ (Art. 10); since E is the middle point of AC, the abscissa of E is $\frac{x'}{2}$ and its ordinate $\frac{y'}{2}$; since F is the middle point of AB, its abscissa is $\frac{a}{2}$ and its ordinate zero. Hence by Art. 35,

the equation to AD is $y = \dfrac{y'x}{x' + a}$ ....(1);

the equation to BE is $y = \dfrac{y'(x - a)}{x' - 2a}$ ....(2);

the equation to CF is $y = \dfrac{y'(2x - a)}{2x' - a}$ ....(3).

To find the point of intersection of (2) and (3) we put

$$\frac{y'(x-a)}{x'-2a} = \frac{y'(2x-a)}{2x'-a};$$

$$\therefore\ (x-a)(2x'-a) = (2x-a)(x'-2a);$$

$$\therefore\ 3ax = a(x'+a);$$

$$\therefore\ x = \tfrac{1}{3}(x'+a).$$

Substitute this value of x in (2) and we find

$$y = \frac{y'}{3}.$$

We have thus determined the co-ordinates of the point of intersection of (2) and (3); moreover we see that these values satisfy (1); hence the line represented by (1) passes through the intersection of the lines represented by (2) and (3), which proves the proposition.

The lines drawn from the angles of a triangle perpendicular to the opposite sides meet in a point.

The equation to BC is (Art. 35)

$$y = \frac{y'}{x' - a}(x - a);$$

hence the equation to the line through A perpendicular to BC is (Art. 44)

$$y = -\frac{x' - a}{y'}x \text{ ....}(4).$$

The equation to AC is

$$y = \frac{y'}{x'}x;$$

hence the equation to the line through B perpendicular to AC is

$$y = -\frac{x'}{y'}(x - a) \text{......................}(5).$$

The line through C perpendicular to AB will be parallel to the axis of y, and its equation will be (Art. 15)

$$x = x' \text{...............................}(6).$$

Now at the point of intersection of (5) and (6) we have

$$x = x', \ y = -\frac{x'}{y'}(x' - a);$$

and as these values satisfy (4), the line represented by (4) passes through the intersection of the lines represented by (5) and (6).

The lines drawn through the middle points of the sides of a triangle respectively perpendicular to those sides meet in a point

The equation to the line through D perpendicular to BC is

$$y - \frac{y'}{2} = -\frac{x' - a}{y'}\left(x - \frac{a + x'}{2}\right) \quad\text{..................(7).}$$

The equation to the line through E perpendicular to CA is

$$y - \frac{y'}{2} = -\frac{x'}{y'}\left(x - \frac{x'}{2}\right) \quad\text{.......................... (8).}$$

The equation to the line through F perpendicular to AB is

$$x = \frac{a}{2} \quad\text{..................................(9).}$$

Now at the point of intersection of (8) and (9) we have

$$x = \frac{a}{2}, \quad y = \frac{y'}{2} - \frac{x'}{y'}\left(\frac{a}{2} - \frac{x'}{2}\right);$$

these values satisfy (7); hence the lines represented by (7), (8), and (9), meet in a point.

Let us denote by P the point of intersection of the three lines in the first proposition, by Q the point of intersection of the three lines in the second proposition, and by R the point of intersection of the three lines in the third proposition; we will now prove that P, Q, and R lie in one straight line. The co-ordinates

of P are $x = \frac{1}{3}(x' + a)$, $\quad y = \frac{y'}{3}$;

of Q are $x = x'$, $\quad y = \frac{x'}{y'}(a - x')$;

of R are $x = \frac{a}{2}$, $\quad y = \frac{y'}{2} - \frac{x'(a - x')}{2y'}$.

Hence the equation to the line passing through P and Q is

$$y - \frac{y'}{3} = \frac{\frac{x'}{y'}(a - x') - \frac{y'}{3}}{x' - \frac{1}{3}(x' + a)}\left(x - \frac{x' + a}{3}\right) \quad\text{...........(10).}$$

In this equation put $x=\frac{a}{2}$, then

$$y-\frac{y'}{3}=\frac{\frac{x'}{y'}(a-x')-\frac{y'}{3}}{\frac{1}{3}(2x'-a)}\left(\frac{a}{6}-\frac{x'}{3}\right)$$

$$=-\tfrac{1}{2}\left\{\frac{x'}{y'}(a-x')-\frac{y'}{3}\right\};$$

$$\therefore\ y=-\frac{x'(a-x')}{2y'}+\frac{y'}{3}+\frac{y'}{6}$$

$$=\frac{y'}{2}-\frac{x'(a-x')}{2y'}.$$

Hence the point R is on the line represented by (10), for the co-ordinates of R satisfy (10).

EXAMPLES.

1. Find the equations to the lines which pass through the following pairs of points:

(1) (0, 1), and (1, −1).
(2) (2, 3), and (2, 4).
(3) (1, 1), and (− 2, − 2).
(4) (0, −a), and (0, −b).

2. Find the equations to the lines which pass through the point (4, 4) and are inclined at an angle of 45° to the line $y=2x$.

3. Find the equations to the lines which pass through the point (0, 1), and are inclined at an angle of 30° to the line $y+x=2$.

4. Find the equations to the lines which pass through the origin and are inclined at an angle of 45° to the line $x=2$.

5. Find the equations to the lines which pass through the origin and are inclined at an angle of 60° to the line $x + y\sqrt{3} = 1$.

6. Find the angle between the lines $x + y = 1$, $y = x + 2$; also find the co-ordinates of the point of intersection.

7. Find the angle between the lines $x + y\sqrt{3} = 0$ and $x - y\sqrt{3} = 2$.

8. What is the angle between $x + 3y = 1$ and $x - 2y = 1$?

9. Find the equations to the lines passing through a given point in the axis of x, and making an angle of 45° with the axis of x.

10. Find the equation to the line which passes through the origin and is perpendicular to the line $x + y = 2$.

11. Find the perpendicular distance of the point $(1, -2)$ from the line $x + y - 3 = 0$.

12. Find the length of the perpendicular from the point (a, b) on the line $\frac{x}{a} + \frac{y}{b} = 1$.

13. Find the co-ordinates of the point of intersection of the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$.

14. Find the equation to the line which passes through the point (a, b), and through the intersection of the lines

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{b} + \frac{y}{a} = 1.$$

15. Shew what loci are represented by the equations

(1) $x^2 + y^2 = 0$, (2) $x^2 - y^2 = 0$,

(3) $x^2 + xy = 0$, (4) $xy = 0$,

(5) $x^2 + y^2 + a^2 = 0$, (6) $x(y - a) = 0$.

16. Interpret

(1) $(x-a)(y-b)=0$,

(2) $(x-a)^2+(y-b)^2=0$,

(3) $(x-y+a)^2+(x+y-a)^2=0$.

17. What straight lines are represented by the equation

$$y^2-4xy+3x^2=0\,?$$

18. Shew that $3y^2-8xy-3x^2+30x-27=0$ represents two straight lines at right angles to one another.

19. Find the equations to the diagonals of the four-sided figure, the sides of which are represented by the equations

$$x=4,\quad y=5,\quad y=x,\quad y=2x.$$

20. $ABCDEF$ is a regular hexagon; take A for the origin, AB as axis of x, and a line through A perpendicular to AB as axis of y; find the equations to all the lines joining the angular points of the hexagon.

21. Given the co-ordinates of the angular points of a triangle, find the equation to the line which joins the middle points of two sides.

22. Find the tangent of the angle between the lines

$$y-mx=0 \text{ and } my+x=0,$$

when referred to oblique axes.

23. Shew that whether the axes be rectangular or oblique the lines $y+x=0$ and $y-x=0$ are at right angles.

24. Given the lengths of two sides of a parallelogram and the angle between them, write down the equations to the two diagonals and find the angle between them; taking one of the corners as origin, and the two sides which meet in that corner as axes.

25. In the figure to Art. 76, take BA and BC as the axes of x and y; suppose $BA=a$, $BC=c$; and let h, k be the co-ordinates of D; then form the equations to AC, BD, AD, CD.

26. With the notation of the preceding question, find the co-ordinates of the middle point of AC and those of the middle point of BD, and form the equation to the line passing through these two points.

27. With the same notation find the co-ordinates of the middle point of EF, and thus shew that this point lies on the line joining the middle points of AC and BD.

28. If $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{a'}+\frac{y}{b'}=1$ be the equations to two lines, which with the co-ordinate axes (rectangular or oblique) contain equal areas, and x', y' be the co-ordinates of the point of their intersection; shew that

$$\frac{y'}{x'}=\frac{b-b'}{a'-a}.$$

29. What points on the axis of x are at a perpendicular distance a from the line $\frac{x}{a}+\frac{y}{b}=1$?

30. Form the equation for determining the abscissa of a point, in the straight line of which the equation is $\frac{x}{a}+\frac{y}{b}=1$, whose distance from a given point (α, β) shall be equal to a given line c. Shew that there are in general two such points, and in the particular case in which those points coincide

$$c^2(a^2+b^2)=(a\beta+b\alpha-ab)^2.$$

31. Find the tangent of the angle between the two lines represented by the equation

$$Ay^2+Bxy+Cx^2=0.$$

32. Find the points of intersection of the straight lines $x+2y-5=0$, $2x+y-7=0$, and $y-x-1=0$; and shew that the area of the triangle formed by them is $\frac{3}{2}$.

33. The area of the triangle formed by the straight lines

$$y=x\tan\alpha,\quad y=x\tan\beta,\quad y=x\tan\gamma+c,$$

is

$$\frac{c^2}{2}\frac{\sin(\alpha-\beta)\cos^2\gamma}{\sin(\alpha-\gamma)\sin(\beta-\gamma)}.$$

34. Given the equations to two parallel straight lines, find the distance between them.

35. Determine the angle between the lines

$$\frac{a}{r} = 4\cos\theta + 3\sin\theta, \qquad \frac{b}{r} = 3\cos\theta - 4\sin\theta.$$

36. Interpret $F(\theta) = 0$; for example, $\sin 3\theta = 1$

37. If the axes be inclined at an angle ω, the condition that the lines

$$Ax + By + C = 0, \qquad A'x + B'y + C' = 0,$$

may be equally inclined to the axis of x in opposite directions is

$$\frac{B}{A} + \frac{B'}{A'} = 2\cos\omega.$$

38. In the preceding question, if besides being equally inclined to the axis of x the lines pass through the origin and are perpendicular to one another, the equation to the straight lines is

$$x^2 + 2xy\cos\omega + y^2\cos 2\omega = 0.$$

39. Two parallel lines are drawn at an inclination θ to the axis of x through the two points whose co-ordinates are a, b, and a', b'; shew that the distance between these lines is $(b'-b)\cos\theta - (a'-a)\sin\theta$. Hence determine the rectangle whose sides pass through four given points, and whose area is given.

40. A square is moved so as always to have the two extremities of one of its diagonals upon two fixed lines at right angles to each other in the plane of the square; shew that the extremities of the other diagonals will at the same time move upon two other fixed straight lines at right angles to each other.

41. AB and BC are two lines perpendicular to each other, A is a fixed point, B moves along a given right line, and AB to BC is a given ratio; determine the locus of C.

42. OX and OY are fixed lines meeting in any angle; a line of given length slides along OX, and another line of given length slides along OY. Find the locus of a point which is so taken that the sum of the areas formed by joining it to the ends of the moving lines is constant.

43. Shew that the lines FC, KB, AL, in the figure to Euclid I. 47, meet in a point.

44. If upon the sides of a triangle as diagonals, parallelograms be described, having their sides parallel to two given lines, the other diagonals of the parallelograms will meet in a point.

45. If from a fixed point O a straight line be drawn $OABCD$... meeting in A, B, C, D,... any given fixed straight lines in one plane, and if

$$\frac{1}{OX}=\frac{1}{OA}+\frac{1}{OB}+\frac{1}{OC}+\dots$$

X being a point in OA, the locus of X is a straight line.

46. Shew that the area of the triangle contained by the axis of y and the lines

$$y=m_1x+c_1, \quad y=m_2x+c_2,$$

is

$$\frac{(c_2-c_1)^2}{2(m_2-m_1)}.$$

47. Determine the area of the triangle contained by the lines

$$y=m_1x+c_1, \quad y=m_2x+c_2, \quad y=m_3x+c_3.$$

48. The area of the triangle formed by the three straight lines

$$y=ax-\frac{bc}{2}, \quad y=bx-\frac{ac}{2}, \quad y=cx-\frac{ab}{2},$$

is

$$\frac{(a-b)(b-c)(c-a)}{8}.$$

CHAPTER IV.

STRAIGHT LINE CONTINUED.

65. We have seen that each of the equations

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0,$$

represents a straight line. We will now interpret the equation

$$Ax + By + C + \lambda (A'x + B'y + C') = 0 \text{ .... } (1),$$

where λ is some constant quantity.

I. Equation (1) must represent *some* straight line, because it is of the first degree in the variables x, y. (Art. 16.)

II. The line represented by (1) passes through the intersection of the lines represented by

$$Ax + By + C = 0 \text{ .... } (2),$$

$$A'x + B'y + C' = 0 \text{ .... } (3).$$

For the values of x and y which satisfy *simultaneously* (2) and (3) will obviously satisfy (1); that is, the point in which (2) and (3) intersect lies on (1).

III. By giving a suitable value to the constant λ the equation (1) may be made to represent *any* straight line which passes through the intersection of (2) and (3).

For let x_1, y_1 denote the co-ordinates of the point of intersection of (2) and (3); suppose *any* line drawn through this point, and let x_2, y_2 be the co-ordinates of another point in it. Now we have already shewn in II. that the line (1) passes through (x_1, y_1); we have therefore only to prove that by giving a suitable value to λ the line (1) can be made to pass

through (x_2, y_2), because two straight lines which have two common points must coincide. Substitute x_2, y_2 for x and y respectively in (1), and determine λ so as to satisfy the equation. Thus

$$\lambda = -\frac{Ax_2 + By_2 + C}{A'x_2 + B'y_2 + C'} \cdot = -\frac{Ax_1 + By_1 + C}{A'x_1 + B'y_1 + C'}$$

Now use this value of λ in (1); then the equation

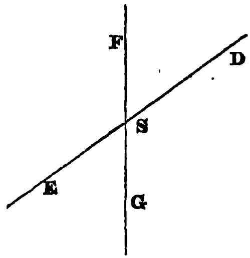
$$Ax + By + C - \frac{Ax_2 + By_2 + C}{A'x_2 + B'y_2 + C'}(A'x + B'y + C') = 0 \ldots (4)$$

represents a straight line passing through (x_1, y_1) and (x_2, y_2).

We have thus proved that by giving a suitable value to λ, the equation (1) will represent *any* straight line passing through the intersection of (2) and (3).

66. The preceding article is very important, and commonly presents difficulties to beginners. The student should not leave it until he is thoroughly familiar with the three propositions which are contained in it. The first proposition is obvious. To prove the second proposition the student may, if he pleases, actually find the values of x and y which satisfy simultaneously $Ax + By + C = 0$, and $A'x + B'y + C' = 0$, and convince himself, by substituting these values, that they *do* satisfy $Ax + By + C + \lambda(A'x + B'y + C') = 0$. There is, however, no necessity for solving the first equations, because it is evident that values of x and y which make $Ax + By + C$ and $A'x + B'y + C'$ vanish simultaneously must make $Ax + By + C + \lambda(A'x + B'y + C')$ vanish, because they make each of the two members of the expression vanish. The third proposition of the preceding article is usually the most difficult—the student is apt to think it needs no demonstration. It may be obvious, however, that by giving different values to λ, different lines are represented, and that we can thus obtain *as many lines as we please*, but this does not shew that we can by a suitable value of λ in (1) represent *any* line passing through the intersection of (2) and (3).

For example, if the straight lines (2) and (3) be DSE and FSG respectively, *it might have happened* that all the lines represented by (1) fell within the angle FSD and none



within FSE. It requires to be proved then that by giving to λ a suitable value in (1) we can obtain the equation to *any* line through S.

67. It is often convenient to denote by a single symbol the expression which we equate to zero in our investigations in this subject; for example, in Art. 51 we have used the symbol α as an abbreviation for $x \cos \alpha + y \sin \alpha - p$. In like manner we may denote such expressions as $Ax + By + C$, $y - mx - c$, $\frac{x}{a} + \frac{y}{b} - 1, \ldots$ by single symbols, as $u, v, \ldots u', \ldots$ Now it will be seen that the demonstration in Art. 65 applies to *any* form of the equation to a straight line as well as to the form $Ax + By + C = 0$ which we have used. Hence the result may be enunciated thus:—if $u = 0$ and $v = 0$ be the equations to two straight lines, and λ a constant quantity, the equation $u + \lambda v = 0$ will represent a straight line passing through the intersection of the two lines; and by giving a suitable value to λ, the equation will represent *any* straight line passing through the intersection of the two lines.

68. If $u = 0$ and $v = 0$ be the equations to two straight lines, then as we have shewn, $u + \lambda v = 0$ will represent a straight line passing through their intersection; it is sometimes convenient to use the more symmetrical form $lu + mv = 0$, where l and m are both constants. It is obvious that what has been said respecting the first form applies to the second; in

fact the second is deducible from the first by writing $\frac{m}{l}$ for λ. It must be remembered throughout this chapter that l, m, n,... λ,... are *constants*, though for shortness we may omit to state it specially in every article.

69. Similarly if $u=0$, $v=0$, $w=0$, be the equations to three straight lines, and l, m, n be constants, the equation

$$lu+mv+nw=0\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

will represent a straight line. Moreover, by giving suitable values to l, m, n we may in general make this equation represent *any* straight line whatsoever. For suppose we wish this equation to represent the straight line passing through (x_1, y_1) and (x_2, y_2). Let u_1, v_1, w_1 denote the values of u, v, w respectively when we put x_1 for x and y_1 for y; and let u_2, v_2, w_2 be the respective values when x_2 and y_2 are put for x and y respectively. Then determine the values of $\frac{m}{l}$ and $\frac{n}{l}$ from the equations

$$lu_1+mv_1+nw_1=0,$$
$$lu_2+mv_2+nw_2=0;$$

suppose we thus find

$$\frac{m}{l}=\frac{\mu}{\lambda}, \quad \frac{n}{l}=\frac{\nu}{\lambda};$$

substitute these values in the equation

$$u+\frac{m}{l}v+\frac{n}{l}w=0,$$

and we obtain

$$u+\frac{\mu}{\lambda}v+\frac{\nu}{\lambda}w=0,$$

or $$\lambda u+\mu v+\nu w=0,$$

which represents the line passing through the points (x_1, y_1) and (x_2, y_2).

We have said above that the equation (1) can *in general* be made to represent any straight line, because there are exceptions which we now proceed to notice.

When the lines represented by $u=0$, $v=0$, and $w=0$ meet in a point, the equation (1) represents a line which necessarily passes through that point. For since the three given lines meet in a point, u, v, and w vanish simultaneously at that point; therefore $lu+mv+nw$ also vanishes at that point, so that the line represented by equation (1) passes through that point.

When the three given lines are parallel the equations $u=0$, $v=0$, $w=0$ will be of the form

$$Ax+By+C_1=0,$$

$$Ax+By+C_2=0,$$

$$Ax+By+C_3=0,$$

and thus equation (1) may be reduced to

$$Ax+By+\frac{C_1+C_2+C_3}{l+m+n}=0,$$

and this equation represents a line parallel to the given lines.

Thus if the three given lines meet in a point or are parallel, equation (1) will not represent *any* straight line; for the line represented by equation (1), in the former case passes through the point in which the given lines meet, and in the latter case is parallel to the given lines.

We may shew that there is no other exception. For the only case in which the general investigation can fail is when λ, μ, and ν all vanish, that is, when

$$\left.\begin{aligned} v_1w_2-v_2w_1&=0\\ w_1u_2-w_2u_1&=0\\ u_1v_2-u_2v_1&=0 \end{aligned}\right\}\text{..................(2).}$$

We shall now prove that when equations (2) are satisfied, the three given lines either all meet in a point or are parallel.

First suppose that the points (x_1, y_1) and (x_2, y_2) are not on any of the three given lines; so that none of the quantities u_1, v_1 u_2, v_2, w_2 vanish.

From the first of equations (2) we have

$$\frac{v_1}{v_2} = \frac{w_1}{w_2};$$

hence by Art. 47 it follows that the ratio of the perpendiculars from (x_1, y_1) and (x_2, y_2) on the line $v = 0$, is the same as the ratio of the perpendiculars from the same points on the line $w = 0$. Hence it will follow geometrically either that the lines $v = 0$ and $w = 0$ are both parallel to the line joining (x_1, y_1) and (x_2, y_2), or else that these three lines meet in a point. Similar results follow from the second of equations (2), and from the third of equations (2). Hence in this case if equations (2) are satisfied, the three given lines either meet in a point or are parallel.

Next suppose that one of the two given points is situated on one of the three given lines; suppose for example that $w_1 = 0$. Then from the first of equations (2) it follows that either $v_1 = 0$ or $w_2 = 0$. Suppose we take $v_1 = 0$. Then from the second and third of equations (2) we deduce either that $u_1 = 0$ or else that $w_2 = 0$ and $v_2 = 0$; in the former case the three given lines all pass through the point (x_1, y_1); in the latter case the lines $v = 0$ and $w = 0$ both pass through the two points (x_1, y_1) and (x_2, y_2), that is, two of the given lines coincide so that all three will reduce either to two intersecting lines or to two parallel lines. Suppose we take $w_2 = 0$ in conjunction with $w_1 = 0$. Then the line $w = 0$ passes through the given points (x_1, y_1) and (x_2, y_2). From the third of equations (2) we have

$$\frac{u_1}{u_2} = \frac{v_1}{v_2};$$

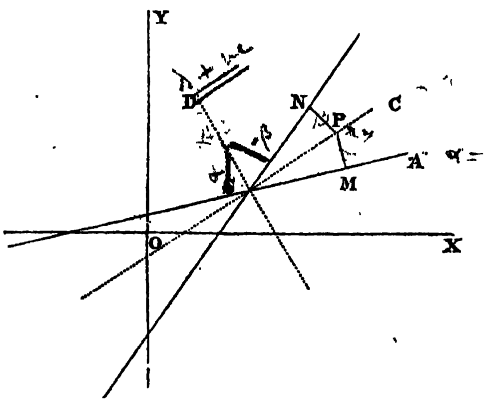
and thus the lines $u = 0$ and $v = 0$ either meet on the line joining the points (x_1, y_1) and (x_2, y_2), or are parallel to this line; that is, the lines $u = 0$, $v = 0$, and $w = 0$ either meet in a point or are parallel.

70. Let $\alpha = 0$, $\beta = 0$ be the equations to two lines expressed in terms of the perpendiculars from the origin and their inclinations to the axis of x (see Art. 50), so that α is an abbreviation for $x \cos \alpha + y \sin \alpha - p_1$, and β is an abbreviation for $x \cos \beta + y \sin \beta - p_2$; we proceed to shew the meaning of the equations $\alpha - \beta = 0$ and $\alpha + \beta = 0$.

Let SA be the line $\alpha = 0$,

SB .... $\beta = 0$;

let SC bisect the angle ASB, and SD bisect the supplement of ASB; the angle DSC is therefore a right angle. Take any point P in SC and draw the perpendiculars PM, PN on SA, SB respectively. If x, y be the co-ordinates of P, the length of PM is α by Art. 54, and the length of PN is β. Since SC bisects the angle ASB, $PM = PN$; therefore for



any point in SC we have $\beta = \alpha$; that is, the equation to SC is

$$\alpha = \beta.$$

Similarly, the equation to SD is

$$\alpha = -\beta.$$

Thus $\alpha-\beta=0$ and $\alpha+\beta=0$ represent the two lines which pass through the intersection of $\alpha=0$ and $\beta=0$ and bisect the angles formed by these lines.

71. The student must distinguish between the lines $\alpha-\beta=0$ and $\alpha+\beta=0$; the following rule may be used: the two lines $\alpha=0$, $\beta=0$, will divide the plane in which they lie into four compartments; ascertain in which of these compartments the origin of co-ordinates is situated; $\alpha-\beta=0$ bisects that angle between $\alpha=0$ and $\beta=0$ in which the origin of co-ordinates lies. This is obvious from the investigation in the preceding article and the remarks in Arts. 53, 54.

The equation $\alpha+\lambda\beta=0$ represents a line such that λ is numerically equal to the ratio of the perpendicular from any point of it on $\alpha=0$ to the perpendicular from the same point on $\beta=0$. If λ is positive the line $\alpha+\lambda\beta=0$ lies in the same two of the four compartments just alluded to as the line $\alpha+\beta=0$; if λ be negative the line $\alpha+\lambda\beta=0$ lies in the same two compartments as the line $\alpha-\beta=0$. From the figure to Art. 70 we see that $PM=PS\sin PSM$ and $PN=PS\sin PSN$; hence λ or $\frac{PM}{PN}=\frac{\sin PSM}{\sin PSN}$; that is, λ expresses the ratio of the sine of the angle between $\alpha=0$ and $\alpha+\lambda\beta=0$ to the sine of the angle between $\beta=0$ and $\alpha+\lambda\beta=0$.

72. We shall continue to express the equation to a straight line by the abbreviation $\alpha=0$ when the equation is of the form $x\cos\alpha+y\sin\alpha-p=0$; when we do not wish to restrict ourselves to this form, we shall use such notation as $u=0$, $v=0$, $u'=0$,......

Let $u=0$, $v=0$ be the equations to two lines, the axes being *rectangular or oblique;* then $u-\lambda v=0$ and $u+\lambda v=0$ represent two lines passing through the intersection of the first two. Suppose, as in Art. 70, that SA, SB are the first two lines and SC, SD the second two; *then will*

$$\frac{\sin CSA}{\sin CSB}=\frac{\sin DSA}{\sin DSB}.$$

For by Art. 57 it appears that if p be the perpendicular from a point (x, y) on the line $u = 0$, then $p = \mu u$, where μ *is a constant quantity;* similarly if p' denote the perpendicular from the same point on $v = 0$, then $p' = \mu' v$, where μ' is a constant quantity. Hence the equation $u - \lambda v = 0$, or $\frac{p}{\mu} - \frac{\lambda p'}{\mu'} = 0$ shews that $\frac{p}{p'} = \frac{\lambda\mu}{\mu'}$; thus we see that numerically without regard to algebraical sign

$$\frac{\sin CSA}{\sin CSB} = \frac{\lambda\mu}{\mu'}.$$

Similarly,
$$\frac{\sin DSA}{\sin DSB} = \frac{\lambda\mu}{\mu'};$$

$$\therefore \frac{\sin CSA}{\sin CSB} = \frac{\sin DSA}{\sin DSB}.$$

73. We will apply the principles of the preceding articles to some examples.

Let $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be the equations to three lines which meet and form a triangle, and suppose the origin of co-ordinates *within* the triangle; then the equations to the three lines bisecting the interior angles of the triangle are, by Art. 70,

$$\beta - \gamma = 0 \ldots (1); \quad \gamma - \alpha = 0 \ldots (2); \quad \alpha - \beta = 0 \ldots (3).$$

These three lines meet in a point; for it is obvious that the values of x and y which simultaneously satisfy (1) and (2) will also satisfy (3).

Again the equations to the three lines which pass through the angles of the triangle and bisect the angles supplemental to those of the triangle are

$$\beta + \gamma = 0 \ldots (4); \quad \gamma + \alpha = 0 \ldots (5); \quad \alpha + \beta = 0 \ldots (6).$$

It is obvious that (3), (4), and (5) meet in a point; similarly (5), (6), and (1) meet in a point; so likewise (4), (6), and (2) meet in a point.

In all our propositions and examples of this kind, we shall always suppose the origin of co-ordinates *within* the triangle, unless the contrary be stated.

74. If $\alpha=0$, $\beta=0$, $\gamma=0$ be the equations to three lines which form a triangle, then *any* line may be represented by an equation of the form $l\alpha+m\beta+n\gamma=0$; for the exceptional cases noticed in Art. 69 cannot occur here.

Let a, b, c denote the lengths of the sides of the triangle which form parts of the lines $\alpha=0$, $\beta=0$, $\gamma=0$ respectively. Take any point within the triangle and join it with the three angular points; thus we obtain three triangles the areas of which are respectively $-\frac{a\alpha}{2}$, $-\frac{b\beta}{2}$, and $-\frac{c\gamma}{2}$. Hence

$$a\alpha+b\beta+c\gamma=\text{a constant};$$

the constant being in fact twice the area of the triangle taken negatively.

This result holds obviously for any point *within* the triangle determined by $\alpha=0$, $\beta=0$, $\gamma=0$. It will be found on examining the different cases which arise that it is also true for any point *without* the triangle. Hence it is universally true.

Suppose we require the equation to a line parallel to the line

$$l\alpha+m\beta+n\gamma=0.$$

This required equation may be written

$$l\alpha+m\beta+n\gamma+k=0,$$

where k is a constant. (Art. 38.)

Or, since $a\alpha+b\beta+c\gamma$ is a constant, the required equation may be written thus,

$$l\alpha+m\beta+n\gamma+k'(a\alpha+b\beta+c\gamma)=0,$$

where k' is a constant.

75. The lines represented by the equations $u=0$, $v=0$, $w=0$, will meet in a point, provided $lu+mv+nw$ is *identically* $=0$; l, m, n being constants. For if $lu+mv+nw=0$ *identically*, we have

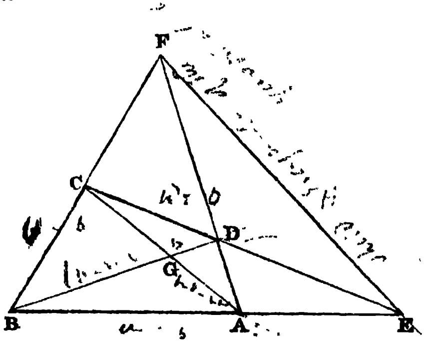
$$w=-\frac{lu+mv}{n} \text{ always.}$$

Hence the equation $w = 0$ may be written

$$-\frac{lu + mv}{n} = 0,$$

that is, the line $w = 0$ is a line passing through the intersection of $u = 0$ and $v = 0$.

76. The following example will furnish a good exercise in the subject.



Let $ABCD$ be a quadrilateral; draw the diagonals AC, BD; produce BA and CD to meet in E, and AD and BC to meet in F; join EF, forming what is called the *third diagonal* of the quadrilateral. Suppose

$u = 0$, the equation to AB, ...(1),

$v = 0$, ... BC, ...(2),

$w = 0$, ... CD, ...(3).

We propose to express the equations to the other lines of the figure in terms of u, v, w, and constant quantities. Assume for the equation to BD

$$lu - mv = 0 \quad \text{..........................} (4),$$

and for the equation to CA

$$mv - nw = 0 \quad \text{........................} (5).$$

These assumptions are legitimate, because (4) represents *some* line passing through B, whatever be the values of the

constants l and m; by properly assuming these constants, we may therefore make (4) represent BD.. Also (5) represents *some* line through C, and by giving a suitable value to n, we may make it represent CA. We may if we please suppose one of the three constants l, m, n, equal to unity, but for the sake of symmetry we will not make this supposition. The equation to AD is

$$lu - mv + nw = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6);$$

for (6) represents a line passing through the intersection of $lu - mv = 0$ and $w = 0$, that is, a line through D; also (6) represents a line passing through the intersection of $u = 0$ and $mv - nw = 0$, that is, a line through A. Hence (6) represents AD. The equation to EF is

$$lu + nw = 0 \ldots\ldots\ldots\ldots\ldots\ldots (7);$$

for (7) obviously represents some line through E, and since $lu + nw = lu - mv + nw + mv$, (7) represents some line through F. Hence (7) represents EF.

Let G be the intersection of AC and BD. The equation to EG is

$$lu - nw = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8);$$

for (8) represents a line passing through the intersection of (1) and (3), and also through the intersection of (4) and (5). The equation to FG is

$$lu - 2mv + nw = 0 \ldots\ldots\ldots\ldots\ldots\ldots (9);$$

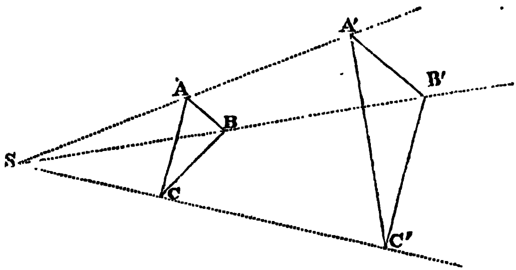
for (9) represents a line passing through the intersection of (4) and (5), and also through the intersection of (2) and (6).

Suppose BD produced to meet EF in H, and AC and EF produced to meet in K; then it may be shewn that the equation to

$$\begin{aligned}
&AH \text{ is } & 2lu - mv + nw &= 0,\\
\text{that to } &CH \text{ is } & mv + nw &= 0,\\
\ldots\ldots\ldots &KB \text{ is } & lu + mv &= 0,\\
\ldots\ldots\ldots &KD \text{ is } & lu - mv + 2nw &= 0.
\end{aligned}$$

We have introduced this example, not on account of any importance in the results, but as an exercise in forming the equations to lines. We proceed to another example.

77. If there be two triangles such that the lines joining the corresponding angles meet in a point, then the intersections of the corresponding sides lie in a straight line.



Let ABC be one triangle, $A'B'C'$ the other triangle; let S be the point in which the lines AA', BB', CC' meet. Let the equation to BC be $u=0$, to CA $v=0$, and to AB $w=0$. Assume for the equation to

$$B'C' \quad l'u+mv+nw=0 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\text{and to } C'A' \quad lu+m'v+nw=0 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

It is shewn in Art. 69 that the equation to $B'C'$ may be written in the above form, and by the method of that article it may be shewn that by giving suitable values to the constants l, m', we may make (2) represent $C'A'$. We will now prove that the equation to $A'B'$ may be written in the form

$$lu+mv+n'w=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The constant n' may be obviously determined, so as to make the line represented by (3) pass through A'; let n' be so determined; it remains to shew that the line (3) will pass through B'. From (1) and (2) it follows that the equation

$$(l'-l)\,u+(m-m')\,v=0 \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

represents *some* line through C'; but (4) obviously represents a line passing through the intersection of BC and CA. Hence (4) is the equation to CC'.

Again, the line represented by (3) by supposition passes through A'; hence from (2) and (3) we see that

$$(m' - m)v + (n - n')\,w = 0 \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

is the equation to AA'.

The equation

$$(l' - l)u + (n - n')\,w = 0 \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

represents a line passing through the intersection of BC and AB, that is, through B; and from (4) and (5) it follows that this line passes through the intersection of CC' and AA', that is, through S. Hence (6) is the equation to SB.

Now from (1) and (3) it follows that the lines represented by these equations meet on the line (6). Hence (3) is the equation to $A'B'$.

The required proposition now easily follows: for the line represented by

$$lu + mv + nw = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

passes through the intersection of BC and $B'C'$, of CA and $C'A'$, and of AB and $A'B'$; that is, these three intersections are in the same straight line.

Conversely, if there be two triangles such that the intersections of the corresponding sides lie in a straight line, then the lines joining the corresponding angles meet in a point. To prove this we may begin with the equations to BC, CA, AB, $B'C'$, $C'A'$ as before, and assume (3) as the equation to *some* line through A'. Then (7) will represent the line passing through the intersection of BC and $B'C'$, and of CA and $C'A'$; now (3) is the equation to a line passing through the intersection of AB and (7); hence (3) must be the equation to $A'B'$. Then from the form of (1), (2), and (3), it follows immediately that CC' passes through the intersection of AA' and BB'.

It may be shewn also that the equation to the line which passes through the intersection of AB and $A'C'$, and of AC and $A'B'$ is

$$lu + m'v + n'w = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

And the intersection of (8) with BC will lie on the line

$$l'u + m'v + n'w = 0 \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

Similarly the line joining the intersection of BA and $B'C'$ with the intersection of BC and $B'A'$ will meet CA on (9). And also the line joining the intersection of CA and $C'B'$ with the intersection of CB and $C'A'$ will meet AB on (9).

78. The equation $u + \lambda v = 0$ represents a straight line passing through the intersection of the lines $u = 0$, $v = 0$. Hence if there be a series of straight lines the equations of which are all of the *form* $u + \lambda v = 0$, and differ merely in having different values of the constant λ, all these lines pass through a point, namely, the intersection of $u = 0$ and $v = 0$.

EXAMPLES.

1. Find the equation to the straight line passing through the origin and the point of intersection of the lines

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad \frac{x}{a'} + \frac{y}{b'} = 1.$$

2. A, A' are two points on the axis of x, and B, B' on that of y, at given distances from the origin; AB and $A'B'$ intersect in P, and AB' and $A'B$ in Q; find the equation to the straight line PQ, and shew that the axes are divided harmonically by it.

3. If $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be the equations to the sides of a triangle ABC opposite the angles A, B, C, prove that $\alpha \sin A - \beta \sin B = 0$ is the equation to the straight line bisecting AB from C.

4. Prove by means of such equations as that given in the preceding question the first proposition in Art. 64.

5. Shew that $\alpha \cos A - \beta \cos B = 0$ is the equation to the perpendicular from C on AB.

6. Hence prove the second proposition in Art. 64.

7. If a, b, c be the lengths of the sides of a triangle opposite the angles A, B, C, respectively, prove that

$$\alpha \cos A - \beta \cos B + \frac{c}{2} (\sin B \cos A - \sin A \cos B) = 0$$

is the equation to the line which bisects AB and is perpendicular to it. The equation may also be written

$$\left(\alpha + \frac{a \sin B \sin C}{2 \sin A}\right) \cos A - \left(\beta + \frac{b \sin C \sin A}{2 \sin B}\right) \cos B = 0.$$

8. Hence prove the third proposition in Art. 64.

9. Interpret the equation $a\alpha + b\beta = 0$.

10. Shew that $a\alpha + b\beta - c\gamma = 0$ is the equation to the line which joins the middle points of AC and BC.

11. Shew that

$$\alpha \cos A + \beta \cos B - \gamma \cos C = 0$$

is the equation to the line which joins the feet of the perpendiculars from A on BC, and from B on AC.

12. If lines be drawn bisecting the angles of a triangle and the exterior angles formed by producing the sides, these lines will intersect in only four points besides the angles of the triangle.

13. If $u = 0$, $v = 0$, $w = 0$ be the equations to three straight lines, find the equation to the line passing through the two points

$$\frac{u}{l} = \frac{v}{m} = \frac{w}{n}, \text{ and } \frac{u}{l'} = \frac{v}{m'} = \frac{w}{n'}.$$

14. Find the equation to the straight line passing through the intersections of the pairs of lines

$$2au + bv + cw = 0, \qquad bv - cw = 0;$$

and

$$2bu + av + cw = 0, \qquad av - cw = 0.$$

15. If $\alpha=0$, $\beta=0$, $\gamma=0$ be the equations to the sides of a triangle ABC, shew that the equation to the straight line which joins the centres of the inscribed circle and the circumscribed circle is

$$\alpha(\cos B-\cos C)+\beta(\cos C-\cos A)+\gamma(\cos A-\cos B)=0.$$

16. If the equations to the sides of a triangle ABC be $u=0$, $v=0$, $w=0$, and to the sides of a triangle $A'B'C'$, $u=a$, $v=b$, $w=c$, then will AA', BB', and CC' meet in a point.

17. If the lines AA', BB', CC', in the last question meet respectively the sides of the triangle ABC in D, E, F, shew that the intersections of DE and AB, of EF and BC, of FD and CA, will all lie in one straight line; and that a similar property will hold for the intersections of the same lines with the sides of the triangle $A'B'C'$.

18. In Art. 76, suppose the line joining F and G to meet AB in P and CD in Q; then find the equations to CP, DP, AQ, BQ, in terms of the notation of that article.

19. From the middle points of the sides of a triangle perpendiculars are drawn (all internal or all external) and proportional to those sides; prove that the straight lines which join the angles with the extremities of the opposite perpendiculars pass through one point.

20. Let the three diagonals of a quadrilateral be produced to meet each other in three points, and let each of these points be joined with the two opposite corners of the quadrilateral; the six lines so drawn will meet each other three and three in four points.

21. In the figure constructed in the preceding question the four lines which meet each other in any corner of the quadrilateral are so related that two of them are parallel to the sides, and two to the diagonals of some parallelogram.

22. Prove that the three points of intersection which are found in questions 4, 6, 8, lie on the straight line

$$\alpha \sin A \cos A \sin (B - C) + \beta \sin B \cos B \sin (C - A) + \gamma \sin C \cos C \sin (A - B) = 0.$$

23. Let any point P be taken in the plane of the triangle ABC, and from the angular points A, B, C, straight lines drawn through it cutting the opposite sides or the sides produced in a, b, c, respectively; let BC, bc be produced to meet in a'; CA and ca in b'; and AB and ab in c'; then shew that the points a', b', c' are in one straight line.

Also prove that the straight lines Bb', Cc', and Aa meet in one point; so also Cc', Aa', and Bb; and Aa', Bb', and Cc.

24. Three points A', B', C' in the sides BC, CA, AB of a triangle being joined form a second triangle of which any two sides make equal angles with the side of the former in which they meet. Shew that AA', BB', CC' are perpendiculars to BC, CA, AB.

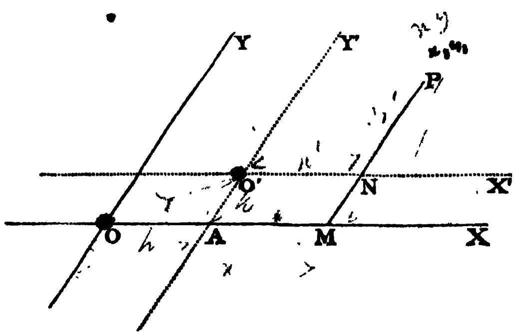
25. ABC is any triangle, O the centre of the inscribed circle, O' the centre of the escribed circle which touches BC. The line OO' meets BC in D, and any straight line drawn through D meets AC in E and AB in F. The lines OF and $O'E$ meet in P, and the lines OE and $O'F$ in Q. Shew that A, P, and Q lie in one straight line perpendicular to OO'.

CHAPTER V.

TRANSFORMATION OF CO-ORDINATES.

79. We have seen in the preceding articles that the *general* equation to a straight line is of the form $y = mx + c$, but that the equation takes more simple forms in particular cases. If the origin is *on the line* the equation becomes $y = mx$; if the axis of x *coincides with the line*, the equation becomes $y = 0$. In a similar manner we shall see as we proceed that the equation to a curve often assumes a more or less simple form, according to the position of the origin and of the axes. It is consequently found convenient to introduce the propositions of the present chapter, which enable us when we know the co-ordinates of a point with respect to any origin and axes, to express the co-ordinates of the same point with respect to any other given origin and axes. It will be seen that these propositions might have been placed at the end of the first chapter, as they involve none of the results of the succeeding chapters.

80. *To change the origin of co-ordinates without changing the direction of the axes, the axes being oblique or rectangular.*



Let OX, OY be the original axes; $O'X'$, $O'Y'$ the new axes; so that $O'X'$ is parallel to OX, and $O'Y'$ to OY. Let h, k be the co-ordinates of O' with respect to O. Let P be any point; x, y its co-ordinates referred to the old axes; x', y' its co-ordinates referred to the new axes.

Let $Y'O'$ produced cut OX in A; draw PM parallel to OY meeting $O'X'$ in N; then

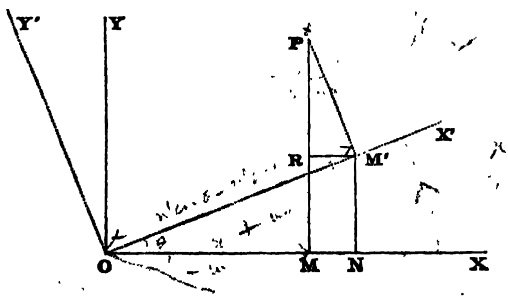
$$OA = h, \qquad AO' = k;$$

$$x = OM = AM + OA = O'N + OA = x' + h,$$

$$y = PM = PN + NM = PN + AO' = y' + k.$$

Hence the old co-ordinates of P are expressed in terms of its new co-ordinates.

81. *To change the direction of the axes without changing the origin, both systems being rectangular.*



Let OX, OY be the old axes; OX', OY' the new axes, both systems being rectangular; let the angle $XOX' = \theta$. Let P be any point; x, y its co-ordinates referred to the old axes; x', y' its co-ordinates referred to the new axes. Draw PM parallel to OY, PM' parallel to OY', $M'N$ parallel to OY, and $M'R$ parallel to OX.

Then $\quad x = OM = ON - MN = ON - M'R$

$$= OM' \cos XOX' - PM' \sin M'PR$$

$$x = x' \cos\theta - y' \sin\theta;$$

$y = x' \sin\theta + y' \cos\theta$

$$y = PM = RM + PR = M'N + PR$$
$$= x' \sin\theta + y' \cos\theta.$$

Hence the old co-ordinates of P are expressed in terms of its new co-ordinates.

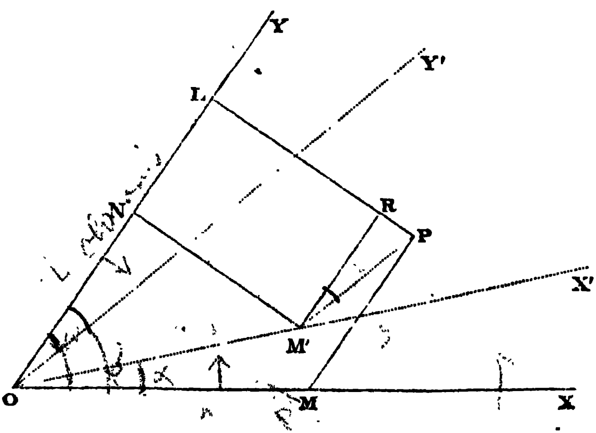
82. In the preceding article θ is measured from the positive part of the axis of x towards the positive part of the axis of y; therefore if in any example to which the formulæ are applied, OX' fall *on the other side* of OX, θ must be considered negative.

From the formulæ of the preceding article, we see that

$$x^2 + y^2 = x'^2 + y'^2;$$

this of course should be the case, since the distance OP is the same whichever system of axes we use.

83. *To change the direction of the axes without changing the origin, both systems being oblique.*



Let OX, OY be the old axes; OX', OY' the new axes. Let (XY) denote the angle between OX, OY; and let a similar notation be used to express the other angles which

are formed by the lines meeting at O. Let P be any point; x, y its co-ordinates referred to the old axes; x', y' its co-ordinates referred to the new axes. Draw PM parallel to OY, and PM' parallel to OY'; from P and M' draw PL, $M'N$ perpendicular to OY; from M' draw $M'R$ perpendicular to PL. Then

$$x = OM, \qquad y = PM;$$
$$x' = OM', \qquad y' = PM'.$$

Now PL = perpendicular from M on $OY = x \sin(XY)$,

also
$$PL = RL + PR = M'N + PR$$
$$= OM' \sin X'OY + PM' \sin Y'OY$$
$$= x' \sin(X'Y) + y' \sin(Y'Y);$$
$$\therefore\ x \sin(XY) = x' \sin(X'Y) + y' \sin(Y'Y) \ldots\ldots\ldots\ldots(1).$$

Similarly by drawing from P and M' perpendiculars on OX we may prove that

$$y \sin(YX) = x' \sin(X'X) + y' \sin(Y'X) \ldots\ldots\ldots(2).$$

Equations (1) and (2) express the old co-ordinates of P in terms of its new co-ordinates; (YX) and (XY) denote the *same angle*, but we use both expressions for greater symmetry.

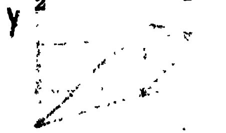
Let $XOX' = \alpha$, $XOY' = \beta$, $XOY = \omega$; then (1) and (2) become

$$x \sin\omega = x' \sin(\omega - \alpha) + y' \sin(\omega - \beta) \ldots\ldots\ldots(3),$$
$$y \sin\omega = x' \sin\alpha \qquad + y' \sin\beta \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

84. Two particular cases of the general proposition in the preceding article may be noticed.

If the original axes are rectangular $\omega = \frac{\pi}{2}$, and the equations (3) and (4) become

$$x = x' \cos\alpha + y' \cos\beta,$$
$$y = x' \sin\alpha + y' \sin\beta.$$



If the new axes be rectangular $\beta = \frac{\pi}{2} + \alpha$, and the equations (3) and (4) become

$$x \sin \omega = x' \sin(\omega - \alpha) - y' \cos(\omega - \alpha),$$
$$y \sin \omega = x' \sin \alpha \qquad + y' \cos \alpha.$$

85. Suppose we require to change both the origin and the direction of the axes; let x, y be the co-ordinates of a point referred to the old axes; x', y' the co-ordinates of the same point referred to the new axes. By Arts. 80 and 83 we have

$$x = x_1 + h,$$
$$y = y_1 + k,$$

where h and k are the co-ordinates of the new origin referred to the old axes, and

$$x_1 = \frac{x' \sin(\omega - \alpha) + y' \sin(\omega - \beta)}{\sin \omega},$$

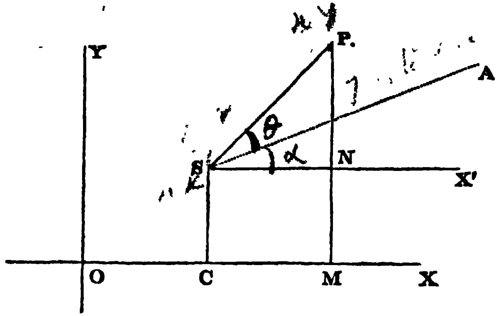
$$y_1 = \frac{x' \sin \alpha + y' \sin \beta}{\sin \omega}.$$

The expressions for x_1 and y_1 will simplify when one or each of the systems is rectangular. (See Art. 84.)

86. The formulæ which connect the rectangular and polar co-ordinates of a point in the particular case in which the origin is the same in both systems, and the axis of x coincides with the initial line, have already been given. (See Art. 8.) The following is the general proposition.

To connect the polar and rectangular co-ordinates of a point.

Let OX, OY be the rectangular axes; let S be the *pole* and SA the *initial* line. Let h, k be the co-ordinates of S referred to O; draw SX' parallel to OX, and let the angle $ASX' = \alpha$.



Let P be any point; x, y its co-ordinates referred to the rectangular axes; r, θ its polar co-ordinates. Draw PM, SC parallel to OY, the former cutting SX' in N, and join SP; then

$$x = OM, \qquad y = PM,$$
$$r = SP, \qquad \theta = \text{the angle } PSA.$$

And $\quad x = OC + CM = OC + SN$

$$= h + r\cos(\theta + \alpha) \ldots\ldots\ldots\ldots(1),$$

$$y = MN + PN = SC + PN$$

$$= k + r\sin(\theta + \alpha) \ldots\ldots\ldots\ldots(2).$$

If $\alpha = 0$ we have

$$x = h + r\cos\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$
$$y = k + r\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

87. By means of the formulæ of the present chapter we shall sometimes be able to simplify the form of an equation; for example, the axes being rectangular, suppose we have

$$y^4 + x^4 + 6x^2y^2 = 2 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This equation represents some locus, and by ascribing different values to x and determining the corresponding values of y from the equation, we can find as many points of the locus as we please. The equation however will be simplified by turning the axes through an angle of 45°. In the formulæ of Art. 81 put $\frac{\pi}{4}$ for θ; thus

$$x = \frac{x' - y'}{\sqrt{2}}, \qquad y = \frac{x' + y'}{\sqrt{2}} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Substitute these values in (1); thus

$$(x' + y')^4 + (x' - y')^4 + 6\,(x'^2 - y'^2)^2 = 8\,;$$

$$\therefore\ 2\,(x'^4 + 6x'^2y'^2 + y'^4) + 6\,(x'^2 - y'^2)^2 = 8,$$

or

$$x'^4 + y'^4 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Since (3) is a simpler form than (1), we shall find it easier to trace the locus by using (3) and the new axes, than by using (1) and the old axes. The student must observe that we make no change in the locus by thus changing the axes or the origin to which we refer it; that is, equation (1) represents precisely the same assemblage of points as (3); for instance, the point for which $x' = 1$ and $y' = 0$ is obviously situated on the locus (3); now *this point* will by (2) have for its co-ordinates referred to the old system

$$x = \frac{1}{\sqrt{2}}, \qquad y = \frac{1}{\sqrt{2}},$$

and these values satisfy (1), that is, *this point* is on the locus (1).

We may remark that we cannot alter the *degree* of an equation by transforming the co-ordinates. For if in the expression $Ax^\alpha y^\beta$ we substitute the values of x and y in terms of x' and y' given in Arts. 80—84, we obtain

$$A\,(ax' + by' + h)^\alpha\,(cx' + ey' + k)^\beta,$$

where a, b, c, e, h, k are all constant quantities; by expanding this expression we shall obtain a series of terms of the form $A'x'^\gamma y'^\delta$, where $\gamma + \delta$ cannot be greater than $\alpha + \beta$. Hence the degree of an equation cannot be *raised* by transformation of co-ordinates. Neither can it be *depressed*; for if from a given equation we could by transformation obtain one of a *lower* degree, then by retracing our steps we should be able from the second equation to obtain one of a *higher* degree, which has been proved to be impossible.

EXAMPLES.

1. Change the equation $r^2 = a^2 \cos 2\theta$ into one between x and y.

2. Shew that the equation $4xy - 3x^2 = a^2$ is changed into $x^2 - 4y^2 = a^2$, if the axes be turned through an angle whose tangent is 2.

3. Transform $\sqrt{x} + \sqrt{y} = \sqrt{c}$ so that the new axis of x may be inclined at 45° to the original axis.

4. The equation to a curve referred to rectangular axes is $y^2 + 4ay \cot \alpha - 4ax = 0$; find its equation referred to oblique axes inclined at an angle α retaining the same axis of x.

5. Shew that the equation $x^2y^2 = a(x^3 + y^3)$ will admit of solution with respect to y' if the axes be moved through an angle of 45°.

6. If x, y be co-ordinates of a point referred to one system of oblique axes, and x', y' the co-ordinates of the same point referred to another system of oblique axes, and

$$x = mx' + ny', \qquad y = m'x' + n'y',$$

shew that

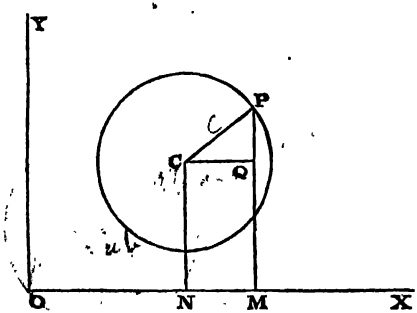
$$\frac{m^2 + m'^2 - 1}{n^2 + n'^2 - 1} = \frac{mm'}{nn'}.$$

CHAPTER VI.

THE CIRCLE.

88. We now proceed to the consideration of the loci represented by equations of the second degree; the simplest of these is the *circle*, with which we shall commence.

To find the equation to the circle referred to any rectangular axes.



Let C be the centre of the circle; P any point on its circumference. Let c be the radius of the circle; a, b the co-ordinates of C; x, y the co-ordinates of P. Draw CN, PM parallel to OY, and CQ parallel to OX. Then

$$CQ^2 + PQ^2 = CP^2;$$

that is, $$(x-a)^2 + (y-b)^2 = c^2 \text{ .... } (1),$$

or $$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - c^2 = 0 \text{ .... } (2).$$

This is the equation required.

The following varieties occur in the equation.

I. Suppose the origin of co-ordinates at the centre of the circle; then $a=0$, and $b=0$; thus (1) and (2) become

$$x^2+y^2-c^2=0 \quad\text{.........................}\quad (3).$$

II. Suppose the origin *on* the circumference of the circle; then the values $x=0$, $y=0$, must satisfy (1) and (2); therefore

$$a^2+b^2-c^2=0,$$

which relation is also obvious from the figure, when O is *on* the circumference; hence (2) becomes

$$x^2+y^2-2ax-2by=0 \quad\text{.................}\quad (4).$$

III. Suppose the origin is *on* the circumference, and that the diameter which passes through the origin is taken for the axis of x; then $b=0$, and $a^2=c^2$; hence (2) becomes

$$x^2+y^2-2ax=0 \quad\text{.....................}\quad (5).$$

Similarly if the origin be on the circumference and the axis of y coincide with the diameter through the origin, we have $a=0$, and $b^2=c^2$; hence (2) becomes

$$x^2+y^2-2by=0 \quad\text{.....................}\quad (6).$$

Hence we conclude from (2) and the following equations, that the equation to a circle when the axes are rectangular is always of the form

$$x^2+y^2+Ax+By+C=0,$$

where A, B, C are constant quantities any one or more of which in particular cases may be equal to zero.

89. We shall next examine, conversely, if the equation

$$x^2+y^2+Ax+By+C=0 \quad\text{...............}\quad (1)$$

always has a circle for its locus.

Equation (1) may be written

$$\left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2=\frac{A^2+B^2}{4}-C \quad\text{..........}\quad (2).$$

I. If A^2+B^2-4C be *negative*, the locus is impossible.

II. If $A^2+B^2-4C=0$, equation (2) represents a *point* the co-ordinates of which are $-\frac{A}{2}$, $-\frac{B}{2}$. This point may be considered as a circle which has an indefinitely small radius.

III. If A^2+B^2-4C be positive, we see by comparing equation (2) with equation (1) of the preceding article that it represents a circle, such that the co-ordinates of its centre are $-\frac{A}{2}$, $-\frac{B}{2}$, and its radius $\frac{1}{2}(A^2+B^2-4C)^{\frac{1}{2}}$.

It will be a useful exercise to construct the circles represented by given equations of the form

$$x^2+y^2+Ax+By+C=0.$$

For example, suppose

$$x^2+y^2+4x-8y-5=0,$$

or $$x+2)^2+(y-4)^2=5+4+16=25.$$

Here the co-ordinates of the centre are -2, 4, and the radius is 5.

Tangent and Normal to a Circle.

90. Def. Let two points be taken on a curve and a secant drawn through them; let the first point remain fixed and the second point move on the curve up to the first; the secant in its limiting position is called the tangent to the curve at the first point.

91. *To find the equation to the tangent at any point of a circle.*

Let the equation to the circle be

$$x^2+y^2=c^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let x', y' be the co-ordinates of the point on the circle at which the tangent is drawn; and x'', y'' the co-ordinates of

an adjacent point on the circle. The equation to the secant through (x', y') and (x'', y'') is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now since (x', y') and (x'', y'') are both on the circumference of the circle,

$$x'^2 + y'^2 = c^2,$$

$$x''^2 + y''^2 = c^2;$$

$\therefore$ by subtraction,

$$x''^2 - x'^2 + y''^2 - y'^2 = 0,$$

or
$$(x'' - x')(x'' + x') + (y'' - y')(y'' + y') = 0;$$

$$\therefore \frac{y'' - y'}{x'' - x'} = -\frac{x'' + x'}{y'' + y'}.$$

Hence (2) may be written

$$y - y' = -\frac{x'' + x'}{y'' + y'}(x - x') \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Now in the limit when (x'', y'') coincides with (x', y'), we have $x'' = x'$, and $y'' = y'$; hence (3) becomes

$$y - y' = -\frac{2x'}{2y'}(x - x') = -\frac{x'}{y'}(x - x').$$

Thus the equation to the tangent at the point (x', y') is

$$y - y' = -\frac{x'}{y'}(x - x') \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

This equation may be simplified; by multiplying by y' and transposing we have

$$xx' + yy' = x'^2 + y'^2;$$

$$\therefore xx' + yy' = c^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

92. The equation to the tangent can be conveniently expressed in terms of the tangent of the angle which the line

makes with the axis of x. For the equation to the tangent at (x', y') is

$$yy' + xx' = c^2,$$

or

$$y = -\frac{x'}{y'}x + \frac{c^2}{y'}.$$

Let $-\frac{x'}{y'} = m$; thus the equation becomes

$$y = mx + \frac{c^2}{y'}.$$

We have then to express $\frac{c^2}{y'}$ in terms of m.

Now $$x' = -my',$$

and $$x'^2 + y'^2 = c^2;$$

$$\therefore\ y'^2 (1 + m^2) = c^2,$$

and $$y' = \frac{c}{\sqrt{(1 + m^2)}}.$$

Hence the equation to the tangent may be written

$$y = mx + c\sqrt{(1 + m^2)}.$$

Conversely every line whose equation is of this form is a tangent to the circle.

93. The definition in Art. 90 may appear arbitrary to the student, and he may ask why we do not adopt that given by Euclid (Def. 2, Book III.). To this we reply that the definition in Art. 90 will be convenient for *every* curve, which is not the case with Euclid's definition. The student however cannot at first be a judge of the necessity or propriety of any definition; he must confine himself to examining the consequences of the definition and the accuracy of the reasoning based upon it.

We may easily shew however that the line represented by the equation

$$xx' + yy' = c^2 \qquad (1)$$

touches, according to Euclid's definition, the circle

$$x^2 + y^2 = c^2 \quad\text{.............................}\quad (2),$$

the point (x', y') being supposed to lie on the circle. To find the point or points of intersection of the line and circle we combine the equations (1) and (2); substitute in (2) the value of y from (1), then

$$x^2 + \left(\frac{c^2 - xx'}{y'}\right)^2 = c^2,$$

or $$x^2 (x'^2 + y'^2) - 2c^2x'x + c^4 - c^2y'^2 = 0,$$

or $$c^2x^2 - 2c^2x'x + c^2x'^2 = 0;$$

$$\therefore\ x^2 - 2xx' + x'^2 = 0;$$

$$\therefore\ x = x';$$

$$\therefore\ \text{from (1), } y = y'.$$

Hence (1) and (2) meet in *only one* point, the point (x', y'). Hence (1) *touches* the circle according to Euclid's definition.

94. Also every line which meets the circle in *one* point only is a tangent to the circle.

For suppose

$$x^2 + y^2 = c^2$$

to be the equation to a circle and

$$y = mx + n$$

the equation to a straight line; to find the points of intersection of the line and circle we combine the equations; thus we obtain

$$(mx + n)^2 + x^2 = c^2$$

or $$(m^2 + 1)\,x^2 + 2mnx + n^2 - c^2 = 0$$

to determine the abscissæ of the points. Now this quadratic equation will have *two* roots except when

$$(m^2 + 1)(n^2 - c^2) = m^2n^2,$$

that is, when $$n^2 = c^2 (1 + m^2).$$

Hence if the straight line meets the circle it must meet it in *two* points unless this condition holds, and then, by Art. 92, the line is a tangent to the circle.

95. Instead of supposing one of the points on the circle fixed and the other to move along the circle as in the definition of Art. 90 we may suppose *both* to move along the circle until they meet at some fixed point of the circle, and the secant in its limiting position will be the tangent at that fixed point. For let (x', y') and (x'', y'') denote the two moving points on the circle, and (x_1, y_1) the fixed point. Then as in equation (3) of Art. 91, we shall have for the equation to the secant

$$y - y' = -\frac{x'' + x'}{y'' + y'}(x - x').$$

In the limit x' and x'' each $= x_1$, and y' and y'' each $= y_1$, and we obtain for the equation to the tangent at (x_1, y_1)

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1),$$

which agrees with the former result.

96. If the equation to a circle be given in the form

$$(x-a)^2 + (y-b)^2 - c^2 = 0,$$

we may find the equation to the tangent at any point in the same manner as in Art. 91.

Let (x', y') be the point on the circle at which the tangent is drawn; (x'', y'') an adjacent point on the circle; then

$$(x'-a)^2 + (y'-b)^2 - c^2 = 0,$$

$$(x''-a)^2 + (y''-b)^2 - c^2 = 0;$$

$$\therefore\ (x''-a)^2 - (x'-a)^2 + (y''-b)^2 - (y'-b)^2 = 0,$$

or $(x''-x')(x''+x'-2a) + (y''-y')(y''+y'-2b) = 0 \ldots (1)$.

Also the equation to the secant through (x', y') and (x'', y'') is

$$y - y' = \frac{y''-y'}{x''-x'}(x - x') \qquad (2).$$

By means of (1) this may be written

$$y-y' = -\frac{x''+x'-2a}{y''+y'-2b}(x-x') \qquad (3).$$

Now in the limit $x''=x'$ and $y''=y'$; hence we have for the equation to the tangent at (x', y')

$$y-y' = -\frac{x'-a}{y'-b}(x-x') \qquad (4).$$

This may be written

$$y-b-(y'-b) = -\frac{x'-a}{y'-b}\{x-a-(x'-a)\};$$

$$\therefore\ (x-a)(x'-a)+(y-b)(y'-b)$$
$$=(x'-a)^2+(y'-b)^2=c^2 \ \ldots\ldots\ldots \ (5).$$

97. Def. The normal at any point of a curve is a straight line drawn through that point perpendicular to the tangent to the curve at that point.

98. *To find the equation to the normal at any point of a circle.*

Let the equation to the circle be

$$x^2+y^2=c^2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1),$$

and let x', y' be the co-ordinates of a point on the circle, then the equation to the tangent at that point is

$$xx'+yy'=c^2,$$

or

$$y=-\frac{x'}{y'}x+\frac{c^2}{y'}.$$

Hence the equation to a line through (x', y') perpendicular to the tangent at that point is

$$y-y'=\frac{y'}{x'}(x-x'),$$

or

$$y=\frac{y'}{x'}x.$$

Since this equation is satisfied by the values $x=0$, $y=0$, the normal at any point passes through the origin of co-ordinates, that is, through the centre of the circle.

99. *From any external point two tangents can be drawn to a circle.*

Let the equation to a circle be

$$x^2+y^2=c^2 \quad\text{.........................}\quad (1),$$

and let h, k be the co-ordinates of an external point. Suppose x', y' the co-ordinates of a point on the circle such that the tangent at this point passes through (h, k). The equation to the tangent at (x', y') is

$$xx'+yy'=c^2 \quad\text{......................}\quad (2).$$

Since this tangent passes through (h, k)

$$hx'+ky'=c^2 \quad\text{......................}\quad (3).$$

Also since (x', y') is on the circle

$$x'^2+y'^2=c^2 \quad\text{....................}\quad (4).$$

Equations (3) and (4) determine the values of x' and y'. Substitute from (3) in (4), thus

$$x'^2+\left(\frac{c^2-hx'}{k}\right)^2=c^2;$$

$$\therefore\ x'^2(h^2+k^2)-2c^2hx'+c^2(c^2-k^2)=0.$$

The roots of this quadratic will be found to be both possible since (h, k) is an *external* point and therefore h^2+k^2 greater than c^2. To each value of x' corresponds one value of y' by (3); hence *two* tangents can be drawn from any external point.

The line which passes through the points where these tangents meet the circle is called the *chord of contact.*

100. *Tangents are drawn to a circle from a given external point; to find the equation to the chord of contact.*

Let h, k be the co-ordinates of the external point; x_1, y_1 the co-ordinates of the point where *one* of the tangents from

(h, k) meets the circle; x_2, y_2 the co-ordinates of the point where the other tangent from (h, k) meets the circle.

The equation to the tangent at (x_1, y_1) is

$$xx_1 + yy_1 = c^2 \text{.....................} \quad (1).$$

Since this tangent passes through (h, k), we have

$$hx_1 + ky_1 = c^2 \text{.....................} \quad (2).$$

Similarly, since the tangent at (x_2, y_2) passes through (h, k),

$$hx_2 + ky_2 = c^2 \text{.....................} \quad (3).$$

Hence it follows that the equation to the *chord of contact* is

$$xh + yk = c^2 \text{.....................} \quad (4).$$

For (4) is obviously the equation to *some straight line;* also this line passes through (x_1, y_1), for (4) is satisfied by the values $x = x_1$, $y = y_1$, as we see from (2); similarly from (3) we conclude that this line passes through (x_2, y_2). Hence (4) is the required equation.

Thus we may proceed as follows in order to draw tangents to a circle from a given external point—draw the line which is represented by (4); join the points where it meets the circle with the given external point and the lines thus obtained are the required tangents.

101. *Through any fixed point chords are drawn to a circle and tangents to the circle drawn at the extremities of each chord the locus of the intersection of the tangents is a straight line.*

Let h, k be the co-ordinates of the point through which the chords are drawn; let tangents to the circle be drawn at the extremities of one of these chords, and let (x_1, y_1) be the point in which they meet. The equation to the corresponding chord of contact is, by Art. 100,

$$xx_1 + yy_1 = c^2.$$

But this chord passes through (h, k); therefore

$$hx_1 + ky_1 = c^2.$$

Hence the point (x_1, y_1) lies on the line

$$xh + yk = c^2;$$

that is, the locus of the intersection of the tangents is a straight line.

We will now prove the converse of this proposition.

102. *If from any point in a straight line a pair of tangents be drawn to a circle, the chords of contact will all pass through a fixed point.*

Let $$Ax + By + C = 0 \quad \text{.....................} \quad (1)$$

be the equation to the straight line; let (x', y') be a point in this line from which tangents are drawn to the circle; then the equation to the corresponding chord of contact is

$$xx' + yy' = c^2 \quad \text{.......................} \quad (2).$$

Since (x', y') is on (1)

$$Ax' + By' + C = 0;$$

therefore (2) may be written

$\therefore y' = -\frac{(Ax'+C)}{B}$

$$xx' - y\frac{Ax' + C}{B} = c^2,$$

or $$\left(x - \frac{Ay}{B}\right)x' - \frac{yC}{B} - c^2 = 0 \quad \text{..............} \quad (3).$$

Now, whatever be the value of x', this line passes through the point whose co-ordinates are found by the simultaneous equations

$$x - \frac{Ay}{B} = 0, \quad \frac{yC}{B} + c^2 = 0;$$

that is, the point for which

$$y = -\frac{Bc^2}{C}, \quad x = -\frac{Ac^2}{C}.$$

103. The student should observe the different interpretations that can be assigned to the equation

$$xh + yk - c^2 = 0.$$

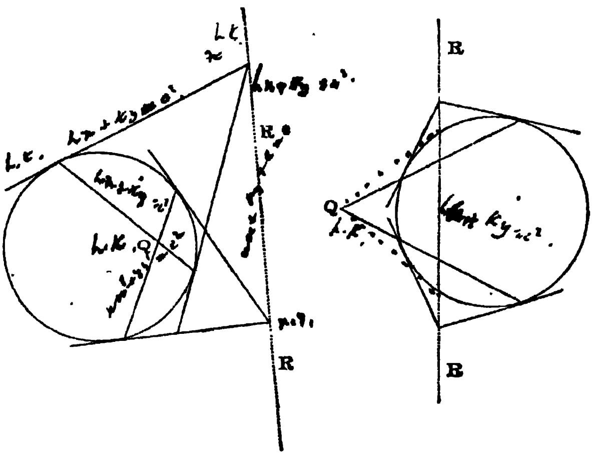
I. If (h, k) be any point whatever, the equation represents the locus of the intersection of tangents at the extremities of each chord through (h, k). (Art. 101.)

II. If (h, k) be an external point, the equation represents the *chord of contact*. (Art. 100.)

III. If (h, k) be *on* the circle, the equation represents the tangent at that point. (Art. 91.)

In the following figures Q denotes the point (h, k), and RR the line

$$xh + yk = c^2.$$



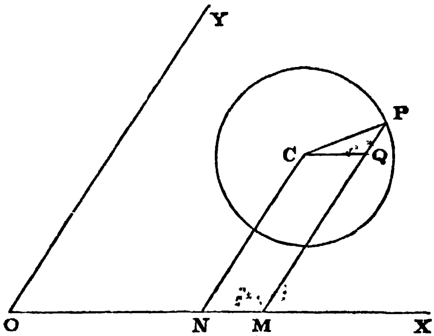
In the first figure Q is *within* the circle, and the line RR receives only the interpretation I.

In the second figure Q is *without* the circle, hence the line RR receives both interpretations I. and II.; if therefore tangents be drawn from Q to the circle ***they will meet it at the points where RR intersects it.***

If Q be *on* the circle, then RR becomes the tangent at Q.

Oblique Axes.

104. *To find the equation to the circle referred to any oblique axes.*



Let ω be the inclination of the axes; let C be the centre of the circle; P any point on its circumference. Let c be the radius of the circle; a, b the co-ordinates of C; x, y the co-ordinates of P. Draw CN, PM parallel to OY, and CQ parallel to OX. Then

$$CP^2 = CQ^2 + PQ^2 - 2CQ \cdot PQ \cos CQP$$
$$= CQ^2 + PQ^2 + 2CQ \cdot PQ \cos \omega;$$

that is, $(x-a)^2 + (y-b)^2 + 2(x-a)(y-b)\cos\omega = c^2$;

or, $x^2 + y^2 + 2xy\cos\omega - 2(a + b\cos\omega)x - 2(b + a\cos\omega)y$
$$+ a^2 + b^2 + 2ab\cos\omega - c^2 = 0.$$

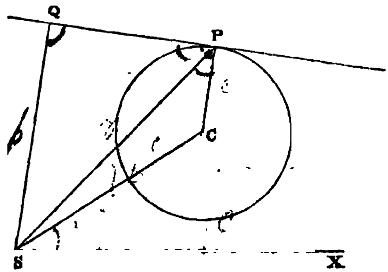
Hence the equation to the circle referred to oblique axes is of the form

$$x^2 + y^2 + 2xy\cos\omega + Ax + By + C = 0,$$

where A, B, C are constant quantities.

Polar Equation.

105. *To find the polar equation to the circle.*



Let S be the pole, SX the initial line; C the centre of the circle, P any point on its circumference.

Let $SC = l$, $CSX = \alpha$, so that l, α are the polar co-ordinates of C; let c be the radius of the circle; and let r, θ be the polar co-ordinates of P.

Then $\qquad CP^2 = PS^2 + CS^2 - 2PS \, . \, CS \, . \cos PSC;$

that is, $\qquad c^2 = r^2 + l^2 - 2lr \cos (\theta - \alpha) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$

or $\qquad r^2 - 2rl (\cos \alpha \cos \theta + \sin \alpha \sin \theta) + l^2 - c^2 = 0 \ \ldots\ (2).$

Hence the polar equation to the circle is of the form

$$r^2 + Ar \cos \theta + Br \sin \theta + C = 0 \quad\ldots\ldots\ldots\ldots\ (3).$$

The polar equation may also be deduced from the equation referred to rectangular axes in Art. 88, by putting $r \cos \theta$ and $r \sin \theta$ for x and y respectively.

pass through the centre

If the initial line ~~be a diameter~~ we have $\alpha = 0$, hence (1) becomes

$$r^2 - 2lr \cos \theta + l^2 - c^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ (4).$$

If, in addition, the origin be on the circumference $l^2 = c^2$,

$$\therefore r = 2l \cos \theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (5).$$

106. *To express the perpendicular from the origin on the tangent at any point in terms of the radius vector of that point.*

Let SQ be the perpendicular from the origin on the tangent at P, and suppose $SQ = p$; then

$$SC^2 = SP^2 + PC^2 - 2SP \cdot PC \cos SPC$$

$$= SP^2 + PC^2 - 2SP \cdot PC \sin SPQ;$$

that is, $$l^2 = r^2 + c^2 - 2cp.$$

In the figure S and C are on the same side of the tangent at P. If we take P so that the tangent at P falls between S and C, we shall find

$$l^2 = r^2 + c^2 + 2cp.$$

107. These equations are sometimes useful in the solution of problems, or demonstration of properties of the circle. For example, take the equation (4) in Art. (105),

$$r^2 - 2rl \cos\theta + l^2 - c^2 = 0;$$

by the theory of quadratic equations we see that the product of the two values of r corresponding to any value of θ is $l^2 - c^2$, which is independent of θ. This agrees with Euclid III. 35, 36.

Also the sum of the two values of r is $2l \cos\theta$; hence if a line be drawn through the pole at an inclination θ to the initial line, the polar co-ordinates of the middle point of the chord which the circle cuts off from this line are

$$\frac{2l\cos\theta}{2}, \text{ and } \theta; \text{ that is, } l\cos\theta, \text{ and } \theta.$$

Hence the polar equation to the locus of the middle point of the chord is

$$r = l\cos\theta,$$

which by (5) in Art. 105, is a circle, of which the diameter is l.

EXAMPLES. 17

1. Determine the position and magnitude of the circles

$$(1) \quad x^2+y^2+4y-4x-1=0,$$

$$(2) \quad x^2+y^2+6x-3y-1=0.$$

2. Find the points of intersection of the circle

$$y^2+x^2=25$$

with the lines

$$y+x=-1, \quad y+x=-5, \quad \text{and} \quad 3y+4x=-25.$$

3. A circle passes through the origin and intercepts lengths h and k respectively from the positive parts of the axes of x and y; determine the equation to the circle.

4. A circle passes through the points (h, k) and (h', k'); shew that its centre must lie on the line

$$(h-h')\left(x-\frac{h+h'}{2}\right)+(k-k')\left(y-\frac{k+k'}{2}\right)=0.$$

5. On the line joining (x', y') and (x'', y'') as diameter a circle is described; find its equation.

6. A and B are two fixed points, and P a point such that $AP=mBP$, where m is a constant; shew that the locus of P is a circle, except when $m=1$.

7. The locus of the point from which two given unequal circles subtend equal angles is a circle. 4

8. Find the equation which determines the points of intersection of the line

$$\frac{x}{h}+\frac{y}{k}-1=0,$$

and the circle

$$x^2+y^2-2ax-2by=0.$$

Deduce the relation that must hold in order that the line may *touch* the circle.

9. Find the equation to the tangent at the origin to the circle

$$x^2 + y^2 - 2y - 3x = 0.$$

10. Shew that the length of the common chord of the circles whose equations are

$$(x-a)^2 + (y-b)^2 = c^2, \qquad (x-b)^2 + (y-a)^2 = c^2,$$

is

$$\sqrt{\{4c^2 - 2(a-b)^2\}}.$$

11. A point moves so that the sum of the squares of its distances from the four sides of a square is constant; shew that the locus of the point is a circle.

12. A point moves so that the sum of the squares of its distances from the sides of an equilateral triangle is constant; shew that the locus of the point is a circle.

13. A point moves so that the sum of the squares of its distances from any given number of fixed points is constant; shew that the locus is a circle.

14. Shew what the equation to the circle becomes when the origin is a point on the perimeter, and the axes are inclined at an angle of 120°, and the parts of them intercepted by the circle are h and k.

15. What must be the inclination of the axes that the equation

$$x^2 + y^2 - xy - hx - hy = 0$$

may represent a circle? Determine the position and magnitude of the circle.

16. What must be the inclination of the axes that the equation

$$x^2 + y^2 + xy - hx - hy = 0$$

may represent a circle? Determine the position and magnitude of the circle.

17. Determine the equation to the circle which has its centre at the origin, and its radius $= 3$, the axes being inclined at an angle of 45°.

18. Determine the equation to the circle which has each of the co-ordinates of its centre $= -\frac{1}{3}$ and its radius $= \dfrac{2}{\sqrt{3}}$, the axes being inclined at an angle of 60°.

19. The axes being inclined at an angle ω, find the radius of the circle

$$x^2 + y^2 + 2xy \cos \omega - hx - ky = 0.$$

20. Shew that the equation to a circle of radius c referred to two tangents inclined at an angle ω as axes is

$$x^2 + y^2 + 2xy \cos \omega - 2(x + y)\, c \cot \frac{\omega}{2} + c^2 \cot^2 \frac{\omega}{2} = 0.$$

21. Shew that the equation in the preceding question may also be written

$$x + y - 2\sqrt{(xy)} \sin \frac{\omega}{2} = c \cot \frac{\omega}{2}.$$

22. Find the value of c in order that the circles

$$(x-a)^2 + (y-b)^2 = c^2, \quad \text{and} \quad (x-b)^2 + (y-a)^2 = c^2,$$

may touch each other.

23. ABC is an equilateral triangle; take A as origin, and AB as axis of x; find the rectangular equation to the circle which passes through A, B, C. Deduce the polar equation to this circle.

24. If the centre of a circle be the pole, shew that the polar equation to the chord of the circle which subtends an angle 2β at the centre is

$$r = c \cos \beta \sec (\theta - \alpha),$$

where α is the angle between the initial line and the line from the centre which bisects the chord. Deduce the polar equation to a line touching the circle at a given point.

25. Find the polar equation to the circle, the origin being on the circumference and the initial line a tangent. Shew

that with this origin and initial line, the polar equation to the tangent at the point θ' is

$$r \sin (2\theta' - \theta) = 2c \sin^2 \theta'.$$

26. Shew that if the origin be on the circumference of a circle, and the diameter through that point make an angle α with the initial line, the equation to the circle is

$$r = 2c \cos (\theta - \alpha).$$

27. Determine the locus of the equation

$$r = A \cos (\theta - \alpha) + B \cos (\theta - \beta) + C \cos (\theta - \gamma) + \ldots\ldots$$

28. AB is a given straight line; through A two indefinite straight lines are drawn equally inclined to AB, and any circle passing through A and B meets those lines in L, M; shew that the *sum* of AL and AM is equal to a constant quantity when L and M are on *opposite* sides of AB, and that the *difference* of AL and AM is constant when L and M are on the *same* side of AB.

29. ABC is an equilateral triangle, and

$$PA = PB + PC,$$

find the locus of P.

30. There are n given straight lines making with another fixed straight line angles $\alpha, \beta, \gamma, \ldots\ldots$; a point P is taken such that the sum of the squares of the perpendiculars from it on these n lines is constant; find the conditions that the locus of P may be a circle.

31. A point moves so that the sum of the squares of its distances from the sides of a regular polygon is constant; shew that the locus of the point is a circle.

32. A line moves so that the sum of the perpendiculars AP, BQ, from the fixed points A and B is constant; find the locus of the middle point of PQ.

33. O is a fixed point and AB a fixed line; a line is drawn from O meeting AB in P; in OP a point Q is taken so that $OP \cdot OQ = k^2$; find the locus of Q.

34. A line is drawn from a fixed point O, meeting a fixed circle in P; in OP a point Q is taken so that $OP \cdot OQ = k^2$; find the locus of Q.

35. Shew that the equation

$$(hy - kx)^2 = c^2 \{(x-h)^2 + (y-k)^2\}$$

represents the two tangents to the circle,

$$x^2 + y^2 = c^2,$$

which pass through the point (h, k).

36. What is represented by the equation

$$r^2 - ra \cos 2\theta \sec \theta - 2a^2 = 0?$$

37. The polar equation to a circle being $r = 2c \cos \theta$, shew that the equation

$$2c \cos \beta \cos \alpha = r \cos (\beta + \alpha - \theta)$$

represents a chord such that the radii drawn to its extremities from the pole, make angles α, β with the initial line.

38. Tangents to a circle at the points P and Q intersect in T; if the lines joining these points with the extremity of a diameter cut a second diameter perpendicular to the former in the points p, q, t, respectively, shew that

$$pt = qt.$$

CHAPTER VII.

RADICAL AXIS. POLE AND POLAR.

Radical Axis.

108. WE have shewn that the equation to a circle is

$$(x-a)^2+(y-b)^2-c^2=0.$$

We shall write this for abbreviation

$$S=0.$$

If the point (x, y) be not on the circumference of the circle, S is not $=0$; we may in that case give a simple geometrical meaning to S.

I. Let (x, y) be *without* the circle; draw a tangent from (x, y) to the circle; join the point of contact with the centre of the circle (a, b); also join (x, y) with (a, b). Let C represent the point (a, b), Q the point (x, y), and T the point of contact of the tangent. Thus we have a right-angled triangle formed, and since $(x-a)^2+(y-b)^2=QC^2$, it follows that $S=QT^2$; that is, S expresses the square of the tangent from (x, y) to the circle. By Euclid III. 36, the square of the tangent is equal to the rectangle of the segments made by the circle on any straight line drawn from (x, y), and thus S will also express the value of this rectangle.

II. Let (x, y) be *within* the circle; then S is negative. Let C and Q have the same meaning as before, and produce CQ to meet the circle in T and T'; then

$$-S=CT^2-CQ^2=(CT-CQ)(CT+CQ)$$
$$=TQ\,.\,T'Q.$$

Hence by Euclid III. 35, if *any* line PQP' be drawn meeting the circle in P and P', the value of the rectangle $PQ \,.\, P'Q$ is $-S$.

109. Let $\quad S$ denote $(x-a)^2+(y-b)^2-c^2$,

and $\quad S'$ denote $(x-a')^2+(y-b')^2-c'^2$;

so that $\quad S=0 \ldots\ldots\ldots\ldots (1)$, and $S'=0 \ldots\ldots\ldots\ldots (2)$,

are the equations to two circles; we proceed to interpret the equation

$$S-S'=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$S-S'$ contains only the first powers of x and y; therefore $S-S'=0$ is the equation to *some straight line.* Also if values of x and y can be found to satisfy simultaneously (1) and (2), these values will satisfy (3). Hence when the circles represented by (1) and (2) intersect, (3) is the equation to the straight line which joins their *points of intersection.*

Also suppose that from any point in (3), external to both circles, we draw tangents to (1) and (2); then, by Art. 108, these tangents are equal in length. Hence whether (1) and (2) intersect or not, the line (3) has the following property;—*if from any point of it lines be drawn to touch both circles, the lengths of these lines are equal.*

110. An equation of the form

$$A(x^2+y^2)+Bx+Cy+D=0$$

will represent a circle; for after division by A we obtain the ordinary form of the equation to a circle. We shall say that the equation to a circle is in its *simplest form* when the coefficient of x^2 and y^2 is unity.

Def. If $S=0$, $S'=0$, be the equations to two circles in their *simplest forms*, the straight line $S-S'=0$ is called the *radical axis* of the circles.

The axes of co-ordinates may here be rectangular or oblique.

Or we may give a geometrical definition thus. A straight line can always be found such that if from any point of it tangents be drawn to two given circles, these tangents are equal; this line is called the radical axis of the circles.

111. *The three radical axes belonging to three given circles meet in a point.*

Let the equations to the three circles be

$$S_1 = 0 \ldots\ldots (1), \qquad S_2 = 0 \ldots\ldots (2), \qquad S_3 = 0 \ldots\ldots (3).$$

The equations to the radical axes are

$$S_1 - S_2 = 0, \text{ belonging to (1) and (2),}$$
$$S_2 - S_3 = 0, \ldots\ldots\ldots\ldots\ldots \text{ (2) and (3),}$$
$$S_3 - S_1 = 0, \ldots\ldots\ldots\ldots\ldots \text{ (3) and (1).}$$

These three lines meet in a point; since it is obvious that the values of x and y which simultaneously satisfy two of the equations, will also satisfy the third.

112. A large number of inferences may be drawn from the preceding articles by examining the special cases which fall under the general propositions. (See Plücker *Analytisch-Geometrische Entwickelungen*, Vol. I. pp. 49—69.) We notice a few of these respecting the radical axis of two circles.

113. *The radical axis is perpendicular to the line joining the centres of the two circles.*

Let the equations to the circles be

$$(x-a)^2 + (y-b)^2 - c^2 = 0,$$
$$(x-a')^2 + (y-b')^2 - c'^2 = 0;$$

then the equation to the radical axis is

$$(x-a)^2 - (x-a')^2 + (y-b)^2 - (y-b')^2 - c^2 + c'^2 = 0;$$

that is,

$$x(a'-a) + y(b'-b) + \tfrac{1}{2}(a^2 - a'^2 + b^2 - b'^2 - c^2 + c'^2) = 0 \ldots\ldots (1).$$

And the equation to the line joining the centres of the circles is (Art. 35)

$$y - b = \frac{b'-b}{a'-a}(x-a) \qquad (2);$$

(1) and (2) are at right angles by Art. 42.

114. When two circles touch, their radical axis is the common tangent at the point of contact. For the radical axis passes through the common point and is perpendicular to the line joining the centres of the circles.

115. Suppose the radius of one of the circles to become indefinitely small, that is, the circle to become a point; the radical axis then has the following property:—if from any point of the radical axis we draw a line to the given point, and a tangent to the given circle, the line and the tangent will be equal in length.

116. The radical axis of a point and a circle falls *without* the circle, whether the point be *without* or *within* the circle. For if the radical axis met the circle, the co-ordinates of the points of intersection would satisfy the *equation to the point* as well as the equation to the circle. But the equation to the point can be satisfied by no co-ordinates except the co-ordinates of that point; therefore the radical axis cannot meet the circle. If the point be *on* the circle, the radical axis is the tangent to the circle at this point.

117. Suppose *both* circles to become points. Then the lines drawn from any point in the radical axis to the two fixed points are equal in length. Hence the radical axis belonging to two given points is the line which bisects at right angles the distance between the two given points.

118. Suppose in Art. 111 that each circle becomes a point; the theorem proved is then the following:—the perpendiculars drawn from the middle points of the sides of a triangle meet in a point.

119. It is a well-known geometrical problem—*to draw a straight line which shall touch two given circles.* If the circles do not intersect, *four* common tangents can be drawn; two of them will be equally inclined to the line joining the centres, and will intersect on that line *between* the circles; the other two will also be equally inclined to the line joining the centres, and will intersect on that line beyond the smaller circle. These two points of intersection are called *centres of similitude.*

For the equations to the common tangents and for the properties of the centres of similitude, we refer to Salmon's *Conic Sections*.

Pole and Polar.

120. Def. If the equation to a circle be

$$x^2 + y^2 = c^2,$$

and h, k be the co-ordinates of any point, then the line

$$xh + yk = c^2$$

is called the *polar* of the point (h, k) with respect to the given circle, and the point (h, k) is called the *pole* of the line

$$xh + yk = c^2$$

with respect to the given circle.

We may also express our definition thus:—the polar of a given point with respect to a given circle is the straight line whose equation involves the co-ordinates of the given point in the same manner as the equation to the tangent at any point of the circle involves the co-ordinates of the point of contact; and the given point is the *pole* of the line.

This definition might be misunderstood. For the equation to the tangent to a circle at a given point might be expressed in different forms by using the relation which holds between the co-ordinates of the given point by virtue of the equation to the circle. We might for example express the equation to the tangent in terms of *either* of the co-ordinates of the given point alone. But in the above definition we mean that the equation to the tangent is to be in the form which it naturally assumes, involving the co-ordinates of the given point *rationally*.

Or we may define the *polar* of a point by means of the properties which it possesses (Art. 103). The polar of a given point with respect to a given circle is the straight line which is the locus of the intersection of tangents drawn at the extremities of every chord through the given point; and the given point is called the pole of this straight line.

If the given point be without the circle, its polar coincides with the *chord of contact* of tangents drawn from that point.

121. *If one straight line pass through the pole of another straight line, the second straight line will pass through the pole of the first straight line.*

Let (x', y') be the pole of the *first* straight line, and therefore

$$xx' + yy' = c^2 \text{.........................} (1)$$

the equation to the *first* straight line.

Let (x'', y'') be the pole of the *second* straight line, and therefore

$$xx'' + yy'' = c^2 \text{.........................} (2)$$

the equation to the *second* straight line.

Since (1) passes through (x'', y'') we have

$$x''x' + y''y' = c^2;$$

and since this equation holds, (2) passes through (x', y').

122. *The intersection of two straight lines is the pole of the line which joins the poles of those lines.*

Denote the two straight lines by A and B, and the line joining their poles by C; since C passes through the pole of A, therefore, by Art. 121, A passes through the pole of C; similarly B passes through the pole of C; therefore the intersection of A and B is the pole of C.

MISCELLANEOUS EXAMPLES.

1. Find the tangent of the angle between the two straight lines whose intercepts on the axes are respectively a, b, and a', b'.

2. If the straight lines represented by the equation

$$x^2 (\tan^2 \phi + \cos^2 \phi) - 2xy \tan \phi + y^2 \sin^2 \phi = 0,$$

make angles α, β with the axis of x, shew that

$$\tan \alpha \sim \tan \beta = 2.$$

3. One side of a square a corner of which is at the origin makes an angle α with the axis of x; find the equations to the four sides and the two diagonals.

4. Find the equations to the diagonals of the parallelogram formed by the straight lines

$$\frac{x}{a}+\frac{y}{b}=1,\quad \frac{x}{a}+\frac{y}{b}=2,$$

$$\frac{x}{b}+\frac{y}{a}=1,\quad \frac{x}{b}+\frac{y}{a}=2;$$

and shew that they are at right angles to one another.

5. The distance of a point (x_1, y_1) from each of two straight lines which pass through the origin of co-ordinates is δ, prove that the two lines are represented by the equation

$$(x_1y-xy_1)^2=(x^2+y^2)\,\delta^2.$$

6. Find the condition that one of the lines represented by

$$Ay^2+Bxy+Cx^2=0$$

may coincide with one of those represented by

$$ay^2+bxy+cx^2=0.$$

7. If $\alpha=0$, $\beta=0$, $\gamma=0$ be the equations to the three sides of a triangle; and a, b, c be the perpendicular distances between these sides and those of another triangle parallel to them respectively, the line joining the centres of the inscribed circles will be represented by any of the equations

$$\frac{\alpha-\beta}{a-b}=\frac{\beta-\gamma}{b-c}=\frac{\gamma-\alpha}{c-a}.$$

8. Prove that the equation to the straight line passing through the middle point of the side BC of a triangle ABC and parallel to the external bisector of the angle A is

$$\beta+\gamma+\frac{a}{2}(\sin B+\sin C)=0.$$

9. The equation to the line drawn parallel to BC through the centre of the escribed circle which touches BC is

$$(\alpha+\beta)\sin B+(\alpha+\gamma)\sin C=0.$$

10. Find the equations to the lines which pass through the intersection of the lines

$$l\alpha+m\beta+n\gamma=0, \qquad l'\alpha+m'\beta+n'\gamma=0,$$

and bisect the angles between them.

11. If $u=0$, $v=0$, be the equations to two circles, shew that by giving a suitable value to the constant λ, the equation $u+\lambda v=0$ will represent any circle passing through the points of intersection of the given circles.

12. A fixed circle is cut by a series of circles, all of which pass through two given points; shew that the lines which join the points of intersection of the fixed circle with each circle of the series all meet in a point.

CHAPTER VIII.

THE PARABOLA.

123. THERE are three curves which we now proceed to define; we shall then deduce their equations from the definitions, and investigate some of their properties from their equations.

DEF. A *conic section* is the locus of a point which moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line. If this ratio be *unity*, the curve is called a *parabola*, if *less* than unity, an *ellipse*, if *greater* than unity, an *hyperbola*.

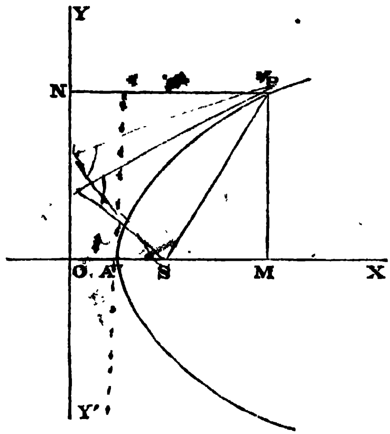
The fixed point is called the focus, and the fixed straight line the directrix.

124. It will be shewn hereafter that if a cone be cut by a plane, the curve of intersection will be one of the following; a parabola, an ellipse, an hyperbola, a circle, two straight lines, one straight line, or a point. Hence the term *conic section* is applied to the parabola, ellipse, and hyperbola—and may be extended to include the circle, two straight lines, one straight line and point. We shall also prove that every curve of the second degree must be a conic section in this larger sense of the term.

At present we confine ourselves to tracing the consequences of the definitions in Art. 123.

125. *To find the equation to the Parabola.*

A parabola is the locus of a point which moves so that its distance from a fixed point is *equal* to its distance from a fixed straight line.



Let S be the fixed point, YY' the fixed straight line. Draw SO perpendicular to YY'; take O as the origin, OS as the direction of the axis of x, OY as that of the axis of y. Suppose $OS = 2a$.

Let P be any point on the locus; join SP; draw PM parallel to OY and PN parallel to OX; let $OM = x$, $PM = y$.

By definition

$$SP = PN;$$

$$\therefore\ SP^2 = PN^2;$$

$$\therefore\ PM^2 + SM^2 = PN^2,$$

that is,

$$y^2 + (x - 2a)^2 = x^2;$$

$$\therefore\ y^2 = 4a(x - a) \quad \text{.............. (1).}$$

This is the equation to the parabola with the assumed origin and axes. The curve cuts the axis of x at a point A which bisects OS; for when $y = 0$ in (1), we have $x = a$. The equation will be simplified if we put the origin at A; let $x' = AM$, then $x' = x - a$, and (1) becomes

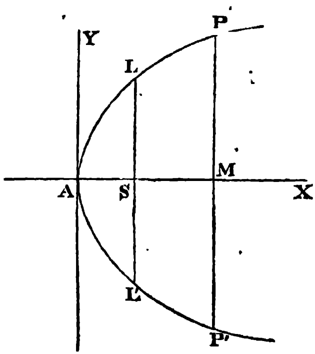
$$y^2 = 4ax'.$$

y^2 4(x − a) (x

We may suppress the accent, if we remember that the origin is now at A; thus we have for the equation to the parabola

$$y^2 = 4ax. \qquad (2).$$

126. *To trace the parabola from its equation* $y^2 = 4ax$.



From this equation we see that for every positive value of x there are *two* values of y, equal in magnitude, but of opposite sign. Hence for every point P on one side of the axis of x, there is a point P' on the other side, such that $P'M = PM$. Hence the curve is symmetrical with respect to the axis of x. Negative values of x do not give possible values of y; hence no part of the curve lies to the left of the origin. As x may have any positive value, the curve extends without limit on the right of the origin.

A is called the *vertex* of the curve and AX the *axis* of the curve.

127. We have drawn the curve *concave* towards the axis of x; the following proposition will justify the figure.

The ordinate of any point of the curve which lies between the vertex and a fixed point of the curve is greater than the corresponding ordinate of the straight line joining the vertex and the fixed point.

Let P be the fixed point; x', y' its co-ordinates; then the equation to AP is

$$y = \frac{y'}{x'} x = \sqrt{\left(\frac{4a}{x'}\right)} \cdot x,$$

since $$y'^2 = 4ax'.$$

Let x denote any abscissa *less than* x', then since the ordinate of the curve is $\sqrt{(4ax)}$, and that of the straight line is $\sqrt{\left(\frac{4a}{x'}\right)} \cdot x$ or $\sqrt{\left(\frac{x}{x'}\right)} \times \sqrt{(4ax)}$, it is obvious that the ordinate of the curve is greater than that of the line.

128. Def. The double ordinate through the focus of a conic section is called the Latus Rectum.

Thus in the figure in Art. 126, LSL' is the Latus Rectum.

Let $x = a$, then from the equation $y^2 = 4ax$, $y = \pm 2a$. Hence $LS = L'S = 2a$; and $LL' = 4a$.

129. *To express the focal distance of any point of the parabola in terms of the abscissa of the point.*

The distance of any point on the curve from the focus is equal to the distance of the same point from the directrix. Hence (see fig. to Art. 125),

$$SP = AM + AS,$$

$$= x + a.$$

Tangent and normal to a Parabola.

130. *To find the equation to the tangent at any point of a parabola.* (See Def. Art. 90.)

Let x', y' be the co-ordinates of the point,

x'', y'' the co-ordinates of an adjacent point on the curve.

The equation to the secant through these points is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x') \dots\dots\dots\dots\dots (1);$$

since (x', y') and (x'', y'') are on the parabola

$$y'^2=4ax', \quad y''^2=4ax'':$$

$$\therefore\ y''^2-y'^2=4a(x''-x');$$

$$\therefore\ \frac{y''-y'}{x''-x'}=\frac{4a}{y''+y'};$$

hence (1) may be written

$$y-y'=\frac{4a}{y''+y'}(x-x').$$

Now in the limit $y''=y'$; hence the equation to the tangent at the point (x', y') is

$$y-y'=\frac{2a}{y'}(x-x') \dots\dots\dots\dots\dots\dots\dots\dots (2).$$

This equation may be simplified; multiply by y', thus

$$\begin{aligned} yy' &= 2a(x-x')+y'^2, \\ &= 2ax-2ax'+4ax', \\ &= 2a(x+x') \dots\dots\dots\dots\dots\dots\dots\dots (3). \end{aligned}$$

131. The equation to the tangent can be conveniently expressed in terms of the tangent of the angle which the line makes with the axis of the parabola.

For the equation to the tangent at (x', y') is

$$yy'=2a(x+x'),$$

or

$$\begin{aligned} y &= \frac{2a}{y'}x+\frac{2ax'}{y'} \\ &= \frac{2a}{y'}x+\frac{4ax'}{2y'} \\ &= \frac{2a}{y'}x+\frac{y'}{2} \dots\dots\dots\dots\dots\dots\dots\dots (1). \end{aligned}$$

Let $$\frac{2a}{y'} = m; \quad \therefore \frac{y'}{2} = \frac{a}{m};$$

thus (1) may be written

$$y = mx + \frac{a}{m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

this is the required equation. Conversely, every line whose equation is of this form is a tangent to the parabola.

132. It may be shewn as in Art. 93, that a tangent to the parabola meets it in only one point. Also, if a line meets a parabola in only one point, it will in general be the tangent at that point.

For suppose

$$y^2 = 4ax \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

to be the equation to a parabola, and

$$y = mx + c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

the equation to a straight line. To determine the abscissæ of the points of intersection, we have the equation

$$(mx + c)^2 = 4ax,$$

or $$m^2x^2 + (2mc - 4a)\,x + c^2 = 0 \ldots\ldots\ldots\ldots (3);$$

this quadratic equation will have two roots, except when

$$(mc - 2a)^2 = m^2c^2,$$

that is, when

$$c = \frac{a}{m}.$$

Hence if the line (2) meets the parabola, it will meet it in two points, unless $c = \frac{a}{m}$, and then the line is a tangent to the parabola by Art. 131.

If, however, the equation (2) be of the form $y = c$, so that the line is parallel to the axis of x, then instead of (3) we have the equation $c^2 = 4ax$, which has but one root; *hence a line*

parallel to the axis of the parabola meets it in only one point, but is not a tangent.

133. The axis of y is a tangent to the curve at the vertex.

For the equation to the tangent at (x', y') is

$$yy' = 2a(x + x');$$

and when $x' = 0$ and $y' = 0$, this becomes

$$x = 0.$$

N 134. *To find the equation to the normal at any point of a parabola.* (See Def. Art. 97.)

Let x', y' be the co-ordinates of the point; the equation to the tangent at that point is

$$y = \frac{2a}{y'}(x + x') \qquad \text{(1)}.$$

The equation to a line through (x', y') perpendicular to (1) is

$$y - y' = -\frac{y'}{2a}(x - x') \qquad \text{(2)}.$$

This is the equation to the normal at (x', y').

135. The equation to the normal may also be expressed in terms of the tangent of the angle which the line makes with the axis of the curve.

For the equation to the normal is

$$y = -\frac{y'}{2a}x + y' + \frac{y'x'}{2a},$$

or

$$y = -\frac{y'}{2a}x + y' + \frac{y'^3}{8a^2} \qquad \text{(1)}.$$

Let $\qquad -\dfrac{y'}{2a} = m; \qquad \therefore\ y' = -2am;$

thus (1) may be written

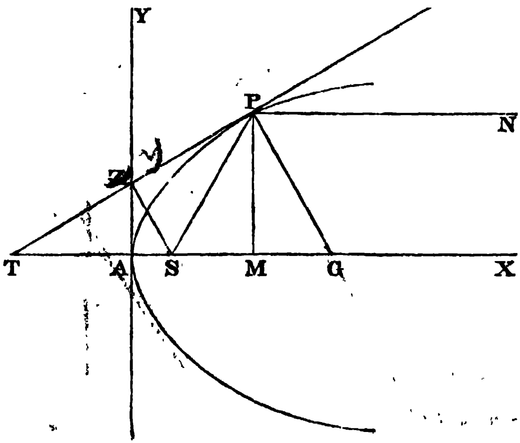
$$y = mx - 2am - am^3 \qquad \text{(2)}.$$

136. We shall now deduce some properties of the parabola from the preceding articles.

Let x', y' be the co-ordinates of P; let PT be the tangent at P and PG the normal at P.

The equation to the tangent at P is

$$yy' = 2a(x + x').$$



Let $y = 0$, then $x = -x'$; hence $AT = AM$.

Also $$ST = AT + AS,$$
$$= AM + AS,$$
$$= SP \text{ (Art. 129)}.$$

Hence the triangle STP is isosceles, and the angle STP = angle SPT.

Thus if PN be parallel to the axis of the curve, PN and PS are equally inclined to the tangent at P.

137. The equation to the normal at P is

$$y - y' = -\frac{y'}{2a}(x - x').$$

At the point G, where the normal cuts the axis, $y=0$; hence from the above equation

$$x-x'=2a;$$

thus $MG=2a=$ half the latus rectum. Also $SG=SP$.

138. *To find the locus of the intersection of the tangent at any point with the perpendicular on it from the focus.*

Let x', y' be the co-ordinates of any point P on the curve; the equation to the tangent at P is

$$y=\frac{2a}{y'}(x+x') \qquad (1).$$

The equation to a line through the focus perpendicular to (1) is

$$y=-\frac{y'}{2a}(x-a) \qquad (2).$$

We have now to eliminate x' and y' by means of (1), (2), and

$$y'^2=4ax' \qquad (3).$$

From (3) we find x' in terms of y', and thus (1) may be written

$$y=\frac{2a}{y'}x+\frac{y'}{2} \qquad (4).$$

Thus the problem is reduced to the elimination of y' from (2) and (4); from (2)

$$y'=-\frac{2ay}{x-a} \qquad (5);$$

substitute in (4); then

$$y=-\frac{(x-a)x}{y}-\frac{ay}{x-a};$$

$$\therefore\ y^2(x-a)+(x-a)^2x+ay^2=0,$$

or

$$\{y^2+(x-a)^2\}x=0 \qquad (6).$$

If the factor $y^2+(x-a)^2$ be equated to zero, we have

$$y=0, \qquad x=a \qquad (7).$$

The point thus determined is the focus; this however is *not* the locus of the intersection of (1) and (2), for the values in (7), although they satisfy (2), do not satisfy (1). We conclude therefore that the required locus is given by the equation

$$x = 0 \qquad (8),$$

which we obtain by considering the other factor in (6).

This result can be easily verified; for if we put $x = 0$ in (1) we obtain $y = \frac{2ax'}{y'} = \frac{y'}{2}$; and if we put $x = 0$ in (2), we also obtain $y = \frac{y'}{2}$; thus (1) and (2) intersect on the line $x = 0$.

Thus, if in the fig. in Art. 136, Z be the intersection of the tangent at P with the axis of y, SZ is perpendicular to the tangent.

139. The process of the preceding article is of frequent use and of great importance. We have in (1) and (2) the equations to two straight lines; if we obtain the values of x and y from these simultaneous equations, we thus determine the point of intersection of the lines; the values of x and y will depend upon those of x' and y', thus giving different points of intersection corresponding to the different lines represented by (1) and (2). If from (1), (2), and (3) we eliminate x' and y' we obtain an equation which holds for the co-ordinates of *every* point of intersection of (1) and (2). This equation is by our definition of a locus the equation corresponding to the locus of the intersection of (1) and (2).

Sometimes the elimination produces, as in the preceding article, an equation which does not represent the required locus. The student has probably noticed in solving algebraical questions that he often arrives at more results than that which he is especially seeking. We can frequently interpret these additional results; thus in the preceding article, since, whatever x' and y' may be, the values $x = a$, $y = 0$, satisfy one of the equations which we use in effecting the elimination, we might anticipate that our result would involve a corresponding factor.

140. If the line from the focus, instead of being perpendicular to the tangent, meet it at any constant angle, the locus of their intersection will still be a straight line. We will indicate the steps of the investigation. Suppose β the angle between the tangent and the line from the focus; equation (1) remains as in Art. 138; instead of (2) we have, by Art. 45,

$$y=\left(\frac{\frac{2a}{y'}+\tan\beta}{1-\frac{2a}{y'}\tan\beta}\right)(x-a)$$

$$=\frac{2a+y'\tan\beta}{y'-2a\tan\beta}(x-a).$$

Instead of (5) in Art. 138, we shall find

$$y'=\frac{2a(x-a)+2ay\tan\beta}{y-(x-a)\tan\beta}.$$

The result of the elimination is

$$y\{y-(x-a)\tan\beta\}\{x-a+y\tan\beta\}$$
$$-x\{y-(x-a)\tan\beta\}^2-a(x-a+y\tan\beta)^2=0.$$

Now, guided by the result of Art. 138, we may anticipate that $y^2+(x-a)^2$ will prove a factor of the left-hand member of the equation; and we shall find by reduction that the equation may be written

$$\{y^2+(x-a)^2\}\{y\tan\beta-x\tan^2\beta-a\}=0.$$

Hence the required locus is

$$y=x\tan\beta+a\cot\beta.$$

141. *To find the length of the perpendicular from the focus on the tangent at any point of the parabola.*

The equation to the tangent at the point (x', y') is

$$y=\frac{2a}{y'}(x+x').$$

The perpendicular on this from the point $(a, 0)$ by Art. 47

$$=\frac{2a(a+x')}{\sqrt{(y'^2+4a^2)}}=\frac{2a(a+x')}{\sqrt{\{4a(a+x')\}}}=\sqrt{\{a(a+x')\}}.$$

Call the focal distance of the point of contact r, and the perpendicular p; then, by Art. 129,

$$r=a+x';$$

$$\therefore\ p=\sqrt{(ar)}.$$

142. *From any external point two tangents can be drawn to a parabola.*

Let the equation to the parabola be

$$y^2=4ax \qquad (1),$$

and let h, k be the co-ordinates of an external point. Suppose x', y' the co-ordinates of a point on the parabola such that the tangent at this point passes through (h, k). The equation to the tangent at (x', y') is

$$yy'=2a(x+x') \qquad (2).$$

Since this tangent passes through (h, k)

$$ky'=2a(h+x') \qquad (3).$$

Also since (x', y') is on the parabola

$$y'^2=4ax' \qquad (4).$$

Equations (3) and (4) determine the values of x' and y'.

Substitute from (4) in (3), thus

$$ky'=2ah+\frac{y'^2}{2},$$

or

$$y'^2-2ky'+4ah=0.$$

The roots of this quadratic will be found to be both possible, since (h, k) is an *external* point and therefore k^2 greater than $4ah$. To each value of y' corresponds one value of x'

by (3); hence *two* tangents can be drawn from any external point.

The line which passes through the points where these tangents meet the parabola is called the *chord of contact.*

143. *Tangents are drawn to a parabola from a given external point; to find the equation to the chord of contact.*

Let h, k be the co-ordinates of the external point; x_1, y_1 the co-ordinates of the point where one of the tangents from (h, k) meets the parabola; x_2, y_2 the co-ordinates of the point where the other tangent from (h, k) meets the parabola.

The equation to the tangent at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \quad \text{.....................} \quad (1).$$

Since this tangent passes through (h, k) we have

$$ky_1 = 2a(h + x_1) \quad \text{.....................} \quad (2).$$

Similarly, since the tangent at (x_2, y_2) passes through (h, k)

$$ky_2 = 2a(h + x_2) \quad \text{.....................} \quad (3).$$

Hence it follows that the equation to the chord of contact is

$$ky = 2a(x + h) \quad \text{.....................} \quad (4).$$

For (4) is obviously the equation to *some* straight line; also this line passes through (x_1, y_1), for (4) is satisfied by the values $x = x_1$, $y = y_1$, as we see from (2); similarly from (3) we conclude that this line passes through (x_2, y_2). Hence (4) is the required equation.

Thus we may proceed as follows in order to draw tangents to a parabola from a given external point. Draw the line which is represented by (4), join the points where it meets the parabola with the given external point, and the lines thus obtained are the required tangents.

144. ***Through any fixed point chords are drawn to a parabola, and tangents to the parabola drawn at the extremities of each chord;—the locus of the intersection of the tangents is a straight line.***

Let h, k be the co-ordinates of the point through which the chords are drawn; let tangents to the parabola be drawn at the extremities of one of these chords, and let (x_1, y_1) be the point in which they meet. The equation to the corresponding chord of contact is, by Art. 143,

$$yy_1 = 2a(x + x_1).$$

But this chord passes through (h, k); therefore

$$ky_1 = 2a(h + x_1).$$

Hence the point (x_1, y_1) lies on the line

$$ky = 2a(x + h);$$

that is, the locus of the intersection of the tangents is a straight line.

We will now prove the converse of this proposition.

145. *If from any point in a straight line a pair of tangents be drawn to a parabola, the chords of contact will all pass through a fixed point.*

Let $$Ax + By + C = 0 \text{ .... } (1)$$

be the equation to the straight line; let (x', y') be a point in this line from which tangents are drawn to the parabola; then the equation to the corresponding chord of contact is

$$yy' = 2a(x + x') \text{ .... } (2).$$

Since (x', y') is on (1)

$$Ax' + By' + C = 0;$$

therefore (2) may be written $y' = -\frac{Ax'+C}{B}$

$$y(Ax' + C) + 2aB(x + x') = 0,$$

or $$(Ay + 2aB)x' + Cy + 2aBx = 0 \text{ .... } (3).$$

Now whatever be the value of x', this line passes through the point whose co-ordinates are found by the simultaneous equations

$$Ay + 2aB = 0, \qquad Cy + 2aBx = 0;$$

that is the point for which

$$y = -\frac{2aB}{A}, \qquad x = \frac{C}{A}.$$

146. The student should observe the different interpretations that can be assigned to the equation

$$ky = 2a(x + h).$$

The statements in Art. 103 with respect to the circle may all be applied to the parabola.

Diameters.

147. *To find the length of a line drawn from any point in a given direction to meet a parabola.*

Let x', y' be the co-ordinates of the point from which the line is drawn; x, y the co-ordinates of the point to which the line is drawn; θ the inclination of the line to the axis of x; r the length of the line; then (Art. 27)

$$x = x' + r\cos\theta, \qquad y = y' + r\sin\theta \text{ .... } (1).$$

If (x, y) be on the parabola, these values may be substituted in the equation $y^2 = 4ax$; thus

$$(y' + r\sin\theta)^2 = 4a(x' + r\cos\theta);$$

$$\therefore\ r^2\sin^2\theta + 2r(y'\sin\theta - 2a\cos\theta) + y'^2 - 4ax' = 0 \text{ ...}(2).$$

From this quadratic two values of r can be found, which are the lengths of the lines that can be drawn from (x', y') in the given direction to the parabola.

When the point (x', y') is *within* the parabola, the roots of the above quadratic will be of *different* signs; in this case the two lines that can be drawn from (x', y') to meet the curve are drawn in *different* directions. When the point (x', y') is *without* the parabola, the roots are of the *same* sign, and the lines are drawn in the *same* direction.

148. Def. *A diameter of a curve is the locus of the middle points of a series of parallel chords.*

149. *To find the diameter of a given system of parallel chords in a parabola.*

Let θ be the inclination of the chords to the axis of the parabola; let x', y' be the co-ordinates of the middle point of any one of the chords; the equation which determines the lengths of the lines drawn from (x', y') to the curve is (Art. 147)

$$r^2 \sin^2 \theta + 2r (y' \sin \theta - 2a \cos \theta) + y'^2 - 4ax' = 0 \text{ ......(1).}$$

Since (x', y') is the *middle* point of the chord, the values of r furnished by this quadratic must be *equal in magnitude* and *opposite in sign;* hence the coefficient of r must vanish; thus

$$y' \sin \theta - 2a \cos \theta = 0;$$

$$\therefore\ y' = 2a \cot \theta \text{ ....(2);}$$

thus the required diameter is a straight line parallel to the axis of the parabola.

Hence every diameter is parallel to the axis of the parabola.

Also every straight line parallel to the axis of the parabola is a diameter, that is, bisects some system of parallel chords; for by giving to θ a suitable value, the equation (2) may be made to represent *any* line parallel to the axis.

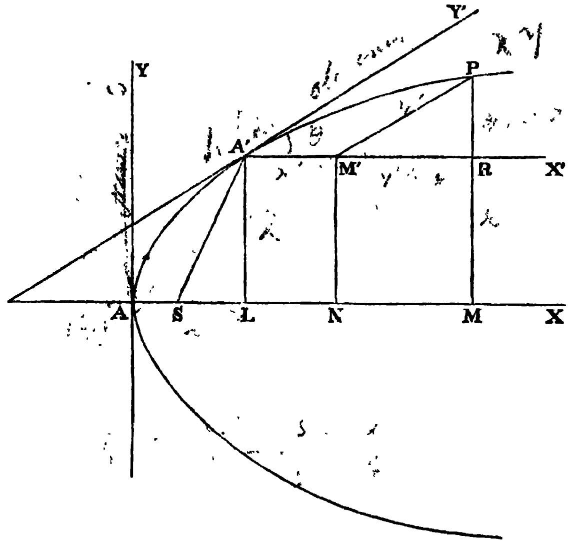
150. Let a tangent be drawn to the parabola at the point where the line $y' = 2a \cot \theta$ meets the parabola; the equation to the tangent is

$$y = \frac{2a}{y'} (x + x');$$

that is, $$y = \tan \theta \, (x + x');$$

hence, the tangent at the extremity of any diameter of the parabola is parallel to the chords which that diameter bisects.

151. *To find the equation to the parabola, the axes being any diameter and the tangent at the point where it meets the curve.*



Let h, k be the co-ordinates of a point A' on the parabola; take this point for a new origin; draw through it a line $A'X'$ parallel to the axis of the curve for the new axis of x, and a tangent $A'Y'$ to the curve for the new axis of y. Let $Y'A'X' = \theta$; then (Art. 150)

$$\frac{2a}{k} = \tan\theta.$$

Let x, y be the co-ordinates of a point P on the curve referred to the original axes; x', y' the co-ordinates of the same point referred to the new axes; draw PM parallel to AY and PM' parallel to $A'Y'$; also draw $A'L$, $M'N$ parallel to AY; let R denote the intersection of PM and $A'X'$; then

$$x = AM = AL + LN + NM = AL + A'M' + M'R$$
$$= h + x' + y'\cos\theta,$$

$$y = PM = RM + PR = A'L + PR$$
$$= k + y' \sin\theta.$$

Substitute these values in the equation $y^2 = 4ax$; thus

$$(k + y' \sin\theta)^2 = 4a\,(h + x' + y' \cos\theta),$$

or $$y'^2 \sin^2\theta + 2y'\,(k \sin\theta - 2a\cos\theta) + k^2 - 4ah = 4ax'.$$

But, $k = 2a \cot\theta$, and $k^2 = 4ah$; thus we have

$$y'^2 \sin^2\theta = 4ax',$$

or $$y'^2 = \frac{4a}{\sin^2\theta}\,x',$$

which is the required equation.

We may prove that

$$\frac{a}{\sin^2\theta} = SA';$$

for $SA' = a + h$ (Art. 129); and

$$h = \frac{k^2}{4a} = a\cot^2\theta;$$

$$\therefore\ a + h = \frac{a}{\sin^2\theta}.$$

Hence the equation may be written

$$y'^2 = 4a'x',$$

where $a' = SA'$; or suppressing the accents on the variables

$$y^2 = 4a'x.$$

152. The equation to the tangent to the parabola will be of *the same form* whether the axes be rectangular, or the oblique system formed by a diameter and the tangent at its extremity; for the investigation of Art. 130 will apply without any change to the equation $y^2 = 4a'x$ which represents a parabola referred to such an oblique system.

153. *Tangents at the extremities of any chord of a parabola meet in the diameter which bisects that chord.*

Refer the parabola to the diameter bisecting the chord, and the corresponding tangent, as axes; let the equation to the parabola be

$$y^2 = 4a'x;$$

let x', y' be the co-ordinates of one extremity of the chord; then the equation to the tangent at this point is

$$yy' = 2a'(x + x') \quad \text{.......................(1)}.$$

The co-ordinates of the other extremity of the chord are x', $-y'$; and the equation to the tangent there is

$$-yy' = 2a'(x + x') \quad \text{.......................(2)}.$$

The lines represented by (1) and (2) meet at the point for which

$$y = 0, \qquad x = -x';$$

this proves the theorem.

Polar Equation.

154. *To find the Polar Equation to the parabola, the focus being the pole.*

Let $SP = r$, $ASP = \theta$, (see Fig. to Art. 125);

then $\quad SP = PN$, by definition;

that is, $\quad SP = OS + SM$;

or $$r = 2a + r\cos(\pi - \theta);$$

$$\therefore\ r(1 + \cos\theta) = 2a,$$

and $$r = \frac{2a}{1 + \cos\theta}.$$

If we denote the angle XSP by θ, then we have as before

$$SP = OS + SM;$$

thus $$r = 2a + r\cos\theta,$$

and $$r = \frac{2a}{1 - \cos\theta}.$$

155. The polar equation to the parabola when the vertex is the pole may be conveniently deduced from the equation $y^2 = 4ax$ by putting $r \cos\theta$ and $r \sin\theta$ for x and y respectively; we thus obtain

$$r = \frac{4a \cos\theta}{\sin^2\theta}.$$

We add a few miscellaneous propositions on the parabola.

DEF. A chord passing through the focus of a conic section is called a focal chord.

156. *If tangents be drawn at the extremities of any focal chord of a parabola,* (1) *the tangents will intersect in the directrix,* (2) *the tangents will meet at right angles,* (3) *the line drawn from the point of intersection of the tangents to the focus will be perpendicular to the focal chord.*

(1) If the tangents to a parabola meet in the point (h, k), the equation to the chord of contact is, by Art. 143,

$$ky = 2a(x + h).$$

Suppose the chord passes through the focus; then the values $x = a$, $y = 0$, must satisfy this equation;

$$\therefore\ 0 = 2a(a + h);$$

$$\therefore\ h = -a;$$

that is, the point of intersection of the tangents is on the directrix.

(2) The equation to the tangent to a parabola may be written (Art. 131)

$$y = mx + \frac{a}{m}.$$

Suppose (h, k) a point on the tangent;

$$\therefore\ hm^2 - km + a = 0.$$

This quadratic will determine the inclinations to the axis of the parabola of the two lines that may be drawn through

the point (h, k) to touch the parabola. Suppose m_1, m_2 the tangents of these inclinations, then by the theory of quadratic equations

$$m_1 m_2 = \frac{a}{h}.$$

If $\quad h = -a, \qquad m_1 m_2 = -1;$

that is, the two tangents are at right angles.

(3) The equation to the line through the focus and (h, k) is

$$y = \frac{k}{h-a}(x-a).$$

If $h = -a$, this becomes

$$y = -\frac{k}{2a}(x-a),$$

and the line is therefore perpendicular to the focal chord of which the equation is

$$yk = 2a(x-a).$$

157. *If through any point within or without a parabola, two lines be drawn parallel to two given straight lines to meet the curve, the rectangles of the segments will be to one another in an invariable ratio.*

Let (x', y') be the given point, and suppose α and β respectively the inclinations of the given straight lines to the axis of the parabola. By Art. 147, if a line be drawn through (x', y') to meet the curve and be inclined at an angle α to the axis, the lengths of its segments are given by the equation

$$r^2 \sin^2\alpha + 2r(y' \sin\alpha - 2a\cos\alpha) + y'^2 - 4ax' = 0.$$

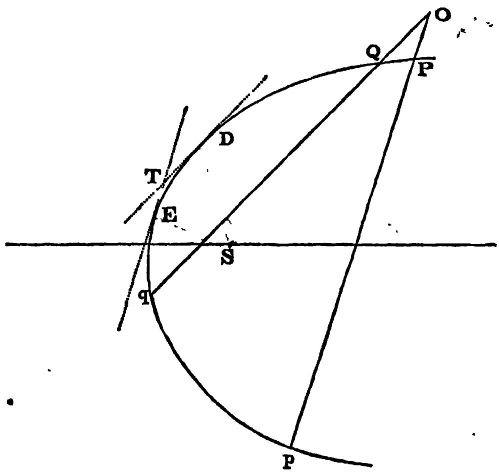
Therefore by the theory of quadratic equations the rectangle of the segments

$$r_1 r_2 = \frac{y'^2 - 4ax'}{\sin^2\alpha}.$$

Similarly the rectangle of the segments of the line drawn through (x', y') at an angle β

$$=\frac{y'^2 - 4ax'}{\sin^2\beta}.$$

Hence the ratio of the rectangles $=\dfrac{\sin^2\beta}{\sin^2\alpha}$,
and this ratio is constant whatever x' and y' may be.



Let O be the point through which the lines OPp, OQq, are drawn inclined to the axis of the parabola at angles α, β, respectively; then we have proved that

$$\frac{OP \cdot Op}{OQ \cdot Oq} = \frac{\sin^2\beta}{\sin^2\alpha}.$$

Let tangents to the parabola be drawn parallel to Pp, Qq, meeting the parabola in E and D respectively; let S be the focus; then by Art. 151,

$$\frac{SE}{SD} = \frac{\sin^2\beta}{\sin^2\alpha}; \quad \therefore \frac{OP \cdot Op}{OQ \cdot Oq} = \frac{SE}{SD}.$$

Suppose O to coincide with T; then $OP \cdot Op$ becomes TE^2 and $OQ \cdot Oq$ becomes TD^2;

$$\therefore \frac{TE^2}{TD^2} = \frac{SE}{SD}.$$

EXAMPLES.

1. Find the equation to the line joining A and L. (See Fig. to Art. 126.)

2. Find the equation to the circle which passes through A, L, L'. (See Fig. to Art. 126.)

3. A point moves so that its shortest distance from a given circle is equal to its distance from a given fixed diameter of that circle; find the locus of the point.

4. Trace the curves $y^2 = 4ax$, and $x^2 + 4ay = 0$; and determine their points of intersection.

5. Determine the equation to the tangent at L. (See Fig. to Art. 126.)

6. Find the angle between the lines in examples 1 and 5.

7. Determine the equation to the normal at L.

8. Find the point where the normal at L meets the curve again, and the length of the intercepted chord.

9. Find the point in a parabola where the tangent is inclined at an angle of 30° to the axis of x.

10. The length of the perpendicular from the foot of the directrix on the tangent at (x', y') is $\dfrac{a(x'-a)}{\sqrt{\{a(x'+a)\}}}$.

11. Find the points of contact of tangents the perpendiculars on which from the foot of the directrix are equal to one-fourth of the latus rectum.

12. A circle has its centre at the vertex A of a parabola whose focus is S, and the diameter of the circle is $3AS$; shew that the common chord bisects AS.

13. Trace the curve $y = x - x^2$, and determine whether the straight line $x + y = 1$ is a tangent to it.

14. The tangent at any point of a parabola will meet the directrix and latus rectum produced in two points equally distant from the focus.

15. PM is an ordinate of a point P in a parabola; a line is drawn parallel to the axis bisecting PM and cutting the curve in Q; MQ cuts the tangent at the vertex A in T; shew that $AT = \frac{2}{3}PM$.

16. If from any point P of a circle PC be drawn to the centre C, and a chord PQ be drawn parallel to the diameter ACB and bisected in R, shew that the locus of the intersection of CP and AR is a parabola.

17. Find the ordinates of the points where the line $y = mx + c$ meets the parabola; hence determine the ordinate of the middle point of the chord which the parabola intercepts on this line.

18. A is the origin, B is a point on the axis of y, BQ a line parallel to the axis of x; in AQ, produced if necessary, P is taken such that its ordinate is equal to BQ; shew that the locus of P is a parabola.

19. From any point Q in the line BQ which is perpendicular to the axis CAB of a parabola whose vertex is A, PQ is drawn parallel to the axis to meet the curve in P; shew that if CA be taken equal to AB, the lines AQ and CP will intersect on the parabola.

20. At the point (x', y') a normal is drawn; find the coordinates of the point where it meets the curve again, and the length of the intercepted chord.

21. If the normal at any point P meet the curve again in Q, and $SP = r$, and p be the perpendicular from S on the tangent at P, then $PQ = \dfrac{4pr}{r-a}$.

22. P is any point on a parabola, A the vertex; through A is drawn a line perpendicular to the tangent at P, and through P is drawn a line parallel to the axis; the lines thus drawn meet in a point Q; shew that the locus of Q is a straight line. Find also the equation to the locus of Q' the intersection of the perpendicular from A and the ordinate at P.

23. PQ is a chord of a parabola, PT the tangent at P. A line parallel to the axis of the parabola cuts the tangent in T, the arc PQ in E, and the chord PQ in F. Shew that

$$TE : EF :: PF : FQ.$$

24. In a parabola whose equation is $y^2 = 4ax$, pairs of tangents are drawn at points whose abscissæ are in the ratio of $1 : \mu$; shew that the equation to the locus of their intersection will be

$$y^2 = (\mu^{\frac{1}{4}} + \mu^{-\frac{1}{4}})^2 ax$$

when the points are on the *same* side of the axis, and

$$y^2 = -(\mu^{\frac{1}{4}} - \mu^{-\frac{1}{4}})^2 ax$$

when they are on *different* sides.

25. Two straight lines are drawn from the vertex of a parabola at right angles to each other; the points where these lines meet the curve are joined, thus forming a right-angled triangle; find the least area of this triangle.

26. Let r and r' be the lengths of two radii vectores drawn at right angles to each other from the vertex of a parabola; then

$$(rr')^{\frac{4}{3}} = 16a^2 (r^{\frac{2}{3}} + r'^{\frac{2}{3}}).$$

27. Find the polar equation to the parabola referred to the foot of the directrix as origin and the axis of the curve as initial line.

28. If a line be drawn from the foot of the directrix cutting the parabola, the rectangle of the intercepts made by the curve is equal to the rectangle of the parts into which the parallel focal chord is divided by the focus.

29. Find the polar equation to the parabola when the foot of the directrix is the origin and the initial line the directrix.

30. A system of parallel chords is drawn in a parabola; find the locus of the point which divides each chord into segments whose product is constant.

31. In a triangle ABC if $\tan A \tan \frac{B}{2} = 2$, and AB be fixed, the locus of C will be a parabola whose vertex is A and focus B.

32. Find the equation to the parabola referred to tangents at the extremities of the latus rectum as axes.

33. Find the equation to the parabola referred to the normal and tangent at L as axes.

34. P is a point on a parabola; x', y' are its co-ordinates; find the equation to the circle described on SP as diameter.

35. Shew that the circle described on SP as diameter *touches* the tangent at the vertex.

36. If the line $y = m(x - a)$ meets the parabola in (x', y') and (x'', y''), shew that

$$x' + x'' = 2a + \frac{4a}{m^2}; \quad x'x'' = a^2; \quad y' + y'' = \frac{4a}{m}; \quad y'y'' = -4a^2.$$

37. A circle is described on a focal chord of a parabola as diameter; if m be the tangent of the inclination of this chord to the axis of x, the equation to the circle is

$$x^2 - 2ax\left(1 + \frac{2}{m^2}\right) + y^2 - \frac{4ay}{m} - 3a^2 = 0.$$

38. Any circle described on a focal chord as diameter *touches* the directrix.

39. If the focus of the parabola be the origin, shew that the equation to the tangent at (x', y') is

$$yy' = 2a\,(x + x' + 2a).$$

40. If the focus of a parabola be the origin, shew that the equation to a tangent to the parabola is

$$y = m\,(x + a) + \frac{a}{m}.$$

41. Two parabolas have a common focus and axis, and a tangent to one intersects a tangent to the other at right angles; find the locus of the point of intersection.

42. If a chord of the parabola $y^2 = 4ax$ be a tangent of the parabola $y^2 = 8a\,(x - c)$, shew that the line $x = c$ bisects that chord.

43. From any point there cannot be drawn more than three normals to a parabola.

44. In a parabola whose equation is $y^2 = 4ax$, the ordinates of three points such that the normals pass through the same point are y_1, y_2, y_3; prove that $y_1 + y_2 + y_3 = 0$. Shew also that a circle described through these three points passes through the vertex of the parabola.

45. If two of the normals which can be drawn to a parabola through a point are at right angles, the locus of that point is a parabola.

46. If two equal parabolas have the same focus and their axes perpendicular to each other, they enclose a space whose *length* PQ = *twice the latus rectum*, and breadth

$$= \frac{\text{latus rectum}}{\sqrt{2}}.$$

47. Find the length of the perpendicular from an external point (h, k) on the corresponding chord of contact.

48. From an external point (h, k) two tangents are drawn to a parabola; shew that the length of the chord of contact is

$$(k^2+4a^2)^{\frac{1}{2}}\,(k^2-4ah)^{\frac{1}{2}}$$

49. From an external point (h, k) two tangents are drawn to a parabola; the area of the triangle formed by the tangents and chord is $\dfrac{(k^2-4ah)^{\frac{3}{2}}}{2a}$

50. Tangents to a parabola TP, Tp are drawn at the extremities of a focal chord; PG, pg are normals at the same points. Shew that $\dfrac{1}{PG^2}+\dfrac{1}{pg^2}$ is invariable; and that the normals subtend equal angles at T.

51. Two equal parabolas have the same axis, but their vertices do not coincide. If through any point O on the inner curve two chords of the outer curve POp, QOq, be drawn at right angles to one another, then $\dfrac{1}{PO\,.\,Op}+\dfrac{1}{QO\,.\,Oq}$ is invariable.

52. A circle described upon a chord of a parabola as diameter just touches the axis; shew that if θ be the inclination of the chord to the axis, $4a$ the latus rectum of the parabola, and c the radius of the circle,

$$\tan\theta = 2a$$

53. If θ, θ' be the inclinations to the axis of the parabola of the two tangents through (h, k), shew that

$$\tan\theta+\tan\theta'=\frac{k}{h};\quad \tan\theta\tan\theta'=\frac{a}{h}.$$

54. If two tangents be drawn to a parabola so that the sum of the angles which they make with the axis is constant,

the locus of their intersection will be a straight line passing through the focus.

55. Shew that the two tangents through (h, k) are represented by the equation

$$h(y-k)^2 - k(y-k)(x-h) + a(x-h)^2 = 0;$$

or $$(k^2 - 4ah)(y^2 - 4ax) = \{ky - 2a(x+h)\}^2.$$

56. Shew that the lines drawn from the vertex to the points of contact of the tangents from (h, k) are represented by the equation

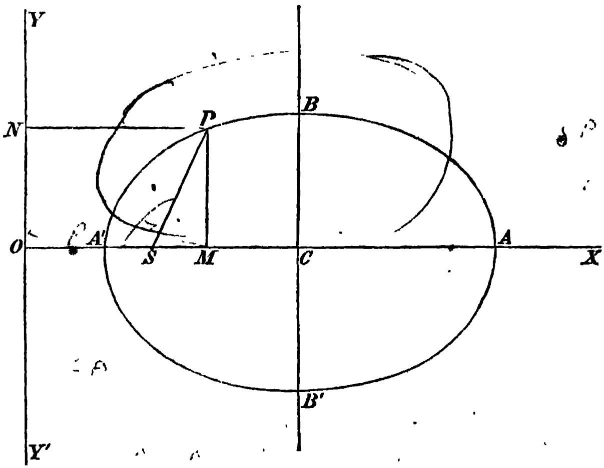
$$hy^2 = 2x(ky - 2ax).$$

CHAPTER IX.

THE ELLIPSE.

158. *To find the equation to the ellipse.*

The ellipse is the locus of a point which moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line, the ratio being less than unity.



Let S be the fixed point, YY' the fixed straight line. Draw SO perpendicular to YY'; take O as the origin, OS as the direction of the axis of x, OY as that of the axis of y.

Let P be a point on the locus; join SP; draw PM parallel to OY and PN parallel to OX. Let $OS = p$, and let e be the ratio of SP to PN. Let x, y be the co-ordinates of P.

By definition,

$$SP = e \, . \, PN;$$

$$\therefore \; SP^2 = e^2 PN^2;$$

$$\therefore \; PM^2 + SM^2 = e^2 PN^2,$$

that is, $$y^2 + (x-p)^2 = e^2 x^2.$$

This is the equation to the ellipse with the assumed origin and axes. [illegible]

159. To find where the ellipse meets the axis of x, we put $y=0$ in the equation to the ellipse; thus

$$(x-p)^2 = e^2 x^2;$$

$$\therefore \; x - p = \pm ex;$$

$$\therefore \; x = \frac{p}{1 \mp e}.$$

Let $OA' = \frac{p}{1+e}$ and $OA = \frac{p}{1-e}$; then A and A' are points on the ellipse.

A and A' are called the *vertices* of the ellipse, and C, the point midway between A and A', is called the *centre* of the ellipse.

160. We shall obtain a simpler form of the equation to the ellipse by transferring the origin to A' or C.

I. Suppose the origin at A'.

Since $OA' = \frac{p}{1+e}$, we put $x = x' + \frac{p}{1+e}$ and substitute this value in the equation

$$y^2 + (x-p)^2 = e^2 x^2;$$

thus $$y^2 + \left(x' + \frac{p}{1+e} - p\right)^2 = e^2 \left(x' + \frac{p}{1+e}\right)^2;$$

or $$y^2 + \left(x' - \frac{ep}{1+e}\right)^2 = e^2 \left(x' + \frac{p}{1+e}\right)^2;$$

$$\therefore\ y^2 + x'^2 - \frac{2x'ep}{1+e} = e^2\left(x'^2 + \frac{2px'}{1+e}\right);$$

$$\therefore\ y^2 = 2pex' - (1-e^2)\,x'^2$$

$$= (1-e^2)\left(\frac{2pex'}{1-e^2} - x'^2\right).$$

The distance $A'A = \frac{p}{1-e} - \frac{p}{1+e} = \frac{2ep}{1-e^2}$, we shall denote this by $2a$; hence the equation becomes

$$y^2 = (1-e^2)\,(2ax' - x'^2).$$

We may suppress the accent if we remember that the origin is at the vertex A', and thus write the equation

$$y^2 = (1-e^2)\,(2ax - x^2) \ldots\ldots\ldots\ldots\ldots (1).$$

II. Suppose the origin at C.

Since $A'C = a$, we put $x = x' + a$ and substitute this value in (1); thus

$$y^2 = (1-e^2)\,\{2a\,(x'+a) - (x'+a)^2\}$$

$$= (1-e^2)\,(a^2 - x'^2).$$

We may suppress the accent if we remember that the origin is now at the centre C, and thus write the equation

$$y^2 = (1-e^2)\,(a^2 - x^2) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In (2) suppose $x = 0$, then $y^2 = (1-e^2)\,a^2$; if then we denote the ordinate CB by b we have $b^2 = (1-e^2)\,a^2$; thus (1) may be written

$$y^2 = \frac{b^2}{a^2}\,(2ax - x^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and (2) may be written

$$y^2 = \frac{b^2}{a^2}\,(a^2 - x^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

or, more symmetrically,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ or } a^2y^2 + b^2x^2 = a^2b^2 \ldots\ldots\ldots (5).$$

161. Since $A'S = e\,OA'$ and $OA' = \dfrac{p}{1+e}$, we have

$$A'S = \frac{ep}{1+e} = \frac{(1-e)\,ep}{1-e^2} = a\,(1-e),$$

$$OA' = \frac{p}{1+e} = \frac{a\,(1-e)}{e},$$

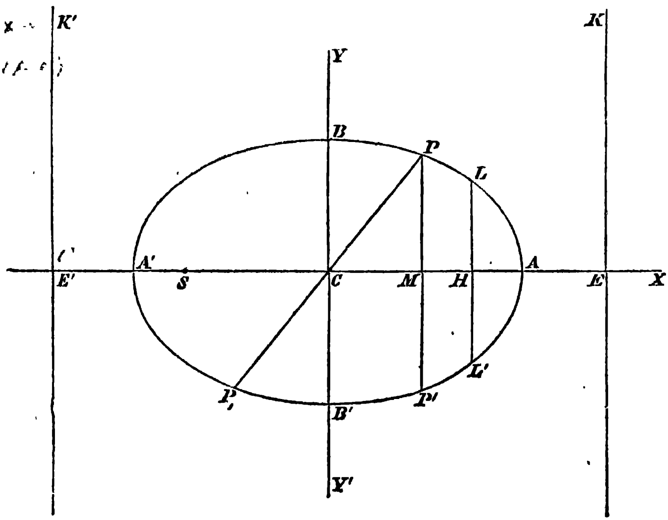
$$SC = A'C - A'S = a - a\,(1-e) = ae,$$

$$OC = A'C + OA' = a + \frac{a\,(1-e)}{e} = \frac{a}{e},$$

$$OS = p = \frac{a\,(1-e^2)}{e}.$$

162. We may now ascertain the form of the ellipse. Take the equation referred to the centre as origin

$$y^2 = \frac{b^2}{a^2}\,(a^2 - x^2). \qquad \text{.(1).}$$



For every value of x less than a there are two values of y, equal in magnitude but of opposite sign. Hence if P be a

point in the curve on one side of the axis of x there is a point P' on the other side of the axis such that $P'M = PM$. Hence the curve is symmetrical with respect to the axis of x. Values of x greater than a do not give possible values of y; hence, CA being equal to a, the curve does not extend to the right of A.

If we ascribe to x any negative value comprised between 0 and $-a$, we obtain for y the same pair of values as when we ascribed to x the corresponding positive value between 0 and a. Hence the portion of the curve to the left of YY' is similar to the portion to the right of YY'.

As the equation (1) may be put in the form

$$x^2 = \frac{a^2}{b^2}(b^2 - y^2) \dots\dots\dots\dots\dots\dots (2),$$

we see that the axis of y also divides the curve symmetrically and that the curve does not extend beyond the points B and B', where CB and CB' each $= b$.

The line $E'K'$ is the directrix; S is the corresponding focus.

Since the curve is symmetrical with respect to the line YCY', it follows that if we take $CH = CS$ and $CE = CE'$, and draw EK perpendicular to CE, the point H and the line EK will form respectively a second focus and directrix by means of which the curve might have been generated.

163. The point C is called the *centre* of the ellipse because *every chord of the ellipse which passes through C is bisected in C.* For suppose (h, k) to be a point on the curve, so that the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is satisfied by the values $x = h$, $y = k$; then $(-h, -k)$ is also a point on the curve, because since $x = h$, $y = k$, satisfy the above equation, it is obvious that $x = -h$, $y = -k$, will also satisfy it. Hence to every point P on the curve there corresponds another point $P_{\prime}$ in the opposite quadrant, such that

$PCP_{,}$ is a straight line and $P_{,}C = PC$. Hence every chord passing through C is bisected in C.

164. We have drawn the curve concave towards the axis of x; the following proposition will justify the figure.

The ordinate of any point of the curve which lies between a vertex and a fixed point of the curve is greater than the corresponding ordinate of the straight line joining that vertex and the fixed point.

Let A' be the vertex, and take it for the origin; let P be the fixed point; x', y' its co-ordinates. Then the equation to the ellipse is (Art. 160)

$$y^2 = \frac{b^2}{a^2}(2ax - x^2).$$

The equation to $A'P$ is $y = \frac{y'}{x'}x$, or $y = \frac{b}{a}\sqrt{\left(\frac{2a}{x'} - 1\right)}\,x$, since (x', y') is on the ellipse.

Let x denote any abscissa less than x', then since the ordinate of the curve is $\frac{b}{a}\sqrt{(2ax - x^2)}$ or $\frac{b}{a}\sqrt{\left(\frac{2a}{x} - 1\right)}\,x$, and that of the straight line is $\frac{b}{a}\sqrt{\left(\frac{2a}{x'} - 1\right)}\,x$, it is obvious that the ordinate of the curve is greater than that of the line.

165. AA' and BB' are called *axes* of the ellipse. The axis AA' which contains the two foci is called the *major* axis and sometimes the *transverse* axis; BB' is called the *minor* axis and sometimes the *conjugate* axis.

The ratio which the distance of any point in the ellipse from the focus bears to the distance of the same point from the corresponding directrix is called the *excentricity* of the ellipse. We have denoted it by the symbol e.

To find the *latus rectum* (see Art. 128) we put $x = CH$, that is, $= ae$, in equation (1) of Art. 162; thus

$$y^2 = \frac{b^2a^2(1 - e^2)}{a^2} = \frac{b^4}{a^2};$$

$$\therefore\ LH = \frac{b^2}{a}, \text{ and the latus rectum } = \frac{2b^2}{a}.$$

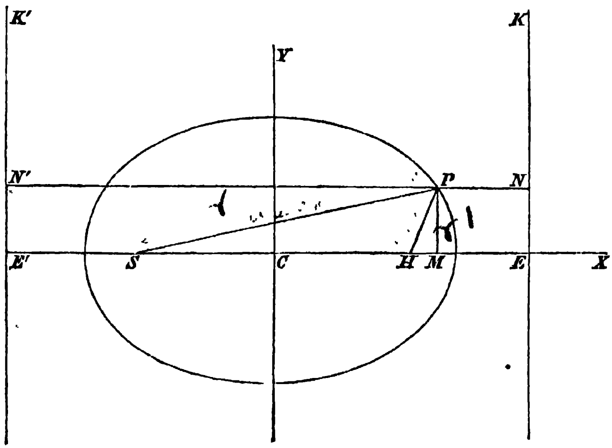
Since $b^2 = a^2 - a^2e^2$; $\therefore\ b^2 + a^2e^2 = a^2$; that is,

$$CB^2 + CH^2 = a^2;$$

$$\therefore\ BH = a;$$

similarly, $$BS = a.$$

166. *To express the focal distances of any point of the ellipse in terms of the abscissa of the point.*



Let S be one focus, $E'K'$ the corresponding directrix; H the other focus, EK the corresponding directrix. Let P be a point on the ellipse; x, y its co-ordinates, the centre being the origin. Join SP, HP, and draw $N'PN$ parallel to the major axis, and PM perpendicular to it.

Then $SP = ePN' = e\,(E'C + CM) = e\left(\dfrac{a}{e} + x\right) = a + ex.$

Also, $HP = ePN = e\,(CE - CM) = e\left(\dfrac{a}{e} - x\right) = a - ex.$

Hence $SP+HP=2a$; that is, the *sum* of the focal distances of any point on the ellipse is equal to the major axis.

167. The equation $y^2=\frac{b^2}{a^2}(a^2-x^2)$ may be written

$$y^2=\frac{b^2}{a^2}(a-x)(a+x).$$

Hence (see Fig. to Art. 162)

$$\frac{PM^2}{A'M.MA}=\frac{BC^2}{AC^2}.$$

168. Let a circle be described on the major axis of the ellipse as a diameter; its equation referred to the centre as origin will be

$$y^2=a^2-x^2.$$

Hence if any ordinate MP of the ellipse be produced to meet the circle in P' we have

$$PM^2=\frac{b^2}{a^2}P'M^2;$$

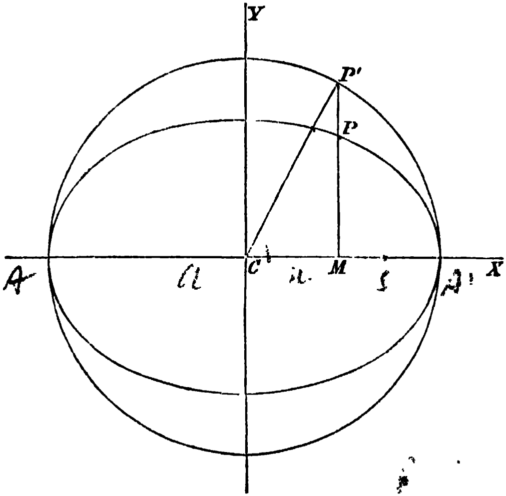
$$\therefore \frac{PM}{P'M}=\frac{b}{a}.$$

Join P' with C the centre of the ellipse; let $P'CM=\phi$, and let x, y be the co-ordinates of P; then

$$x=CP'\cos\phi=a\cos\phi,$$

$$y=\frac{b}{a}P'M=\frac{b}{a}a\sin\phi=b\sin\phi.$$

These values of x and y are sometimes useful in the solution of problems.



The angle $P'CM$ is called the *excentric angle* of the point P.

169. From Art. 160 we see that the equation to the ellipse when the vertex is the origin is

$$y^2 = 2pex - (1 - e^2)\,x^2.$$

If we suppose $e = 1$, this becomes

$$y^2 = 2px,$$

which is the equation to a parabola whose latus rectum is $2p$.

Also in the ellipse

$$a = \frac{ep}{1 - e^2}, \quad b = a\sqrt{(1 - e^2)} = \frac{ep}{\sqrt{(1 - e^2)}},$$

$$AH \text{ or } a\,(1 - e) = \frac{ep}{1 + e}.$$

If we now make $e=1$, we have a and b infinite, and $a(1-e)=\frac{p}{2}$. Thus if we suppose the distance between the vertex and nearer focus of an ellipse to remain constant while the excentricity approaches continually nearer to unity, the major and minor axes of the ellipse increase indefinitely and the ellipse about the vertex approximates to the form of a parabola.

Thus if any property is established for an ellipse we may seek for a corresponding property in the parabola by referring the ellipse to the vertex as origin and examining what the result becomes when e is made to approach continually to unity, while the distance between the vertex and the nearer focus remains constant.

Tangent and Normal to an Ellipse.

170. *To find the equation to the tangent at any point of an ellipse.* (See Def. Art. 90.)

Let x', y' be the co-ordinates of the point,

x'', y'' the co-ordinates of an adjacent point on the curve.

The equation to the secant through these points is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x')\dots\dots\dots\dots\dots\dots(1);$$

since (x', y') and (x'', y'') are points on the ellipse,

$$a^2y'^2+b^2x'^2=a^2b^2,$$

$$a^2y''^2+b^2x''^2=a^2b^2;$$

$$\therefore\ a^2(y''^2-y'^2)+b^2(x''^2-x'^2)=0;$$

$$\therefore\ \frac{y''-y'}{x''-x'}=-\frac{b^2}{a^2}\cdot\frac{x''+x'}{y''+y'}.$$

Hence (1) may be written

$$y - y' = -\frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'} (x - x').$$

Now in the limit $x'' = x'$, and $y'' = y'$; hence the equation to the tangent at the point (x', y') is

$$y - y' = -\frac{b^2 x'}{a^2 y'} (x - x') \ldots\ldots\ldots\ldots \quad (2).$$

This equation may be simplified; multiply by $a^2 y'$, thus

$$a^2 yy' + b^2 xx' = a^2 y'^2 + b^2 x'^2 = a^2 b^2.$$

171. The equation to the tangent can be conveniently expressed in terms of the tangent of the angle which the line makes with the major axis of the ellipse.

For the equation to the tangent at (x', y') is

$$a^2 yy' + b^2 xx' = a^2 b^2,$$

or $$y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'}.$$

Let $-\dfrac{b^2 x'}{a^2 y'} = m$; thus the equation becomes

$$y = mx + \frac{b^2}{y'};$$

we have then to express $\dfrac{b^2}{y'}$ in terms of m.

Now $$b^2 x' = -a^2 y' m,$$

and $$a^2 y'^2 + b^2 x'^2 = a^2 b^2;$$

$$\therefore\ a^2 y'^2 + \frac{a^4 m^2 y'^2}{b^2} = a^2 b^2;$$

$$\therefore\ y'^2 (a^2 m^2 + b^2) = b^4,$$

$$\frac{b^2}{y'} = \sqrt{(a^2 m^2 + b^2)}.$$

Hence the equation to the tangent may be written

$$y = mx + \sqrt{(a^2m^2 + b^2)}.$$

Conversely every line whose equation is of this form is a tangent to the ellipse.

It may be shewn as in Arts. 93, 94, that the tangent at any point of an ellipse meets it in only *one* point, and that a line which meets an ellipse in only one point is the tangent at that point.

172. The tangents at the extremities of either axis are parallel to the other axis.

For the co-ordinates of A are a, 0. (See Fig. to Art. 162.) Hence, putting $x' = a$, $y' = 0$, the equation

$$a^2yy' + b^2xx' = a^2b^2$$

becomes $$x = a,$$

which is the equation to a line through A parallel to CY. Similarly the tangent at A' is parallel to CY, and the tangents at B and B' are parallel to CX.

173. *To find the equation to the normal at any point of an ellipse.* (See Def. Art. 97.)

Let x', y' be the co-ordinates of the point; the equation to the tangent at that point is

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'} \quad\text{.................. (1).}$$

The equation to a line through (x', y') perpendicular to (1) is

$$y - y' = \frac{a^2y'}{b^2x'}(x - x') \quad\text{.................. (2).}$$

This is the equation to the normal at (x', y').

174. The equation to the normal may also be expressed in terms of the tangent of the angle which the line makes with the major axis of the ellipse.

The equation to the normal at (x', y') is

$$y = \frac{a^2 y'}{b^2 x'} x - \left(\frac{a^2}{b^2} - 1\right) y'.$$

Let $\frac{a^2 y'}{b^2 x'} = m$; thus the equation becomes

$$y = mx - \frac{a^2 - b^2}{b^2} y' \qquad \text{.................... (1);}$$

we have then to express $\frac{a^2 - b^2}{b^2} y'$ in terms of m.

Now, $$b^2 x' = \frac{a^2 y'}{m},$$

and $$a^2 y'^2 + b^2 x'^2 = a^2 b^2;$$

$$\therefore\ a^2 y'^2 + \frac{a^4 y'^2}{b^2 m^2} = a^2 b^2;$$

$$\therefore\ y'^2 (b^2 m^2 + a^2) = b^4 m^2.$$

Hence (1) becomes

$$y = mx - \frac{(a^2 - b^2) m}{\sqrt{(b^2 m^2 + a^2)}}.$$

175. We shall now deduce some properties of the ellipse from the preceding articles.

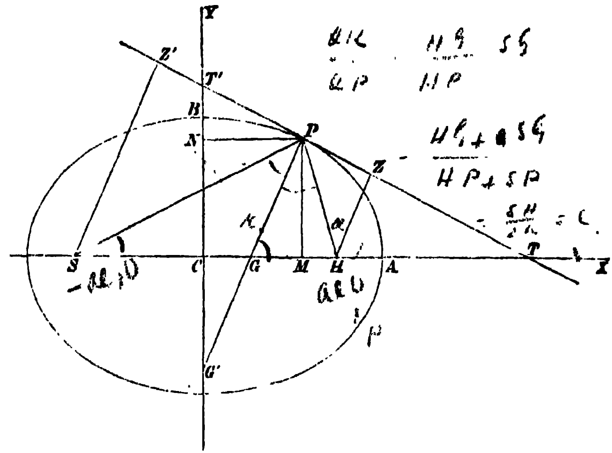
Let x', y' be the co-ordinates of P; let PT be the tangent at P, and PG the normal at P; PM, PN perpendiculars on the axes.

The equation to the tangent at P is

$$a^2 yy' + b^2 xx' = a^2 b^2.$$

Let $y = 0$, then $x = \frac{a^2}{x'}$, hence

$$CT = \frac{CA^2}{CM};$$



$$\therefore\ CM \,.\, CT = CA^2.$$

Similarly, if the tangent at P meet CY in T',

$$CN \,.\, CT' = CB^2.$$

176. The equation to the normal at P is

$$y - y' = \frac{a^2y'}{b^2x'}(x - x').$$

At the point G where the normal cuts the major axis, $y = 0$, hence from the above equation

$$x - x' = -\frac{b^2x'}{a^2};$$

$$\therefore\ x = x'\left(1 - \frac{b^2}{a^2}\right) = e^2x'.$$

Thus $$CG = e^2 CM.$$

At the point G' where the normal cuts the minor axis, $x = 0$, hence from the above equation

$$y = y' - \frac{a^2y'}{b^2} = -\frac{a^2e^2}{b^2}y'.$$

Thus $$CG' = \frac{a^2e^2}{b^2}PM.$$

177. The lengths of PG and PG' may be conveniently expressed in terms of the focal distances of P.

$$PG^2 = PM^2 + GM^2$$
$$= y'^2 + (x' - e^2x')^2$$
$$= y'^2 + x'^2(1-e^2)^2$$
$$= y'^2 + \frac{b^4x'^2}{a^4}$$
$$= \frac{b^2}{a^2}(a^2 - x'^2) + \frac{b^4}{a^4}x'^2$$
$$= \frac{b^2}{a^2}\left\{a^2 - \left(1 - \frac{b^2}{a^2}\right)x'^2\right\}$$
$$= \frac{b^2}{a^2}(a^2 - e^2x'^2).$$

Let $SP = r'$, $HP = r$; then

$$r' = a + ex', \quad r = a - ex';$$

thus
$$PG^2 = \frac{b^2rr'}{a^2}.$$

Similarly, it may be shewn that

$$PG'^2 = \frac{a^2rr'}{b^2}.$$

178. *The normal at any point bisects the angle between the focal distances of that point.*

Let x', y' be the co-ordinates of P; the co-ordinates of S are $-ae$, 0; hence the equation to SP is (Art. 35)

$$y = \frac{y'}{x' + ae}(x + ae) \quad \text{.. (1).}$$

The equation to the normal at P is

$$y - y' = \frac{a^2y'}{b^2x'}(x - x').$$

Hence the tangent of the angle GPS

$$= \frac{\dfrac{a^2y'}{b^2x'} - \dfrac{y'}{x'+ae}}{1 + \dfrac{a^2y'^2}{b^2x'(x'+ae)}} = \frac{(a^2-b^2)\,x'y' + a^2ey'}{a^2y'^2 + b^2x'^2 + b^2x'ae}$$

$$= \frac{a^2e^2x'y' + a^3ey'}{a^2b^2 + b^2x'ae} = \frac{eay'}{b^2}.$$

The equation to HP is

$$y = \frac{y'}{x'-ae}(x-ae);$$

hence it may be shewn that the tangent of the angle GPH also $= \dfrac{eay'}{b^2}$;

$$\therefore\ SPG = HPG.$$

Hence $SPT' = HPT$; that is, the tangent at any point is equally inclined to the focal distances of that point.

179. The preceding proposition may also be established thus:

$$CG = e^2x', \text{ (Art. 176)};$$

$$\therefore\ SG = ae + e^2x',$$

and $$HG = ae - e^2x'.$$

Also $SP = a + ex'$, $HP = a - ex'$; hence

$$\frac{SG}{HG} = \frac{SP}{HP};$$

therefore by Euclid, VI. 3, PG bisects the angle SPH.

180. *To find the locus of the intersection of the tangent at any point with the perpendicular on it from the focus.*

Let $$y = mx + \sqrt{(b^2 + m^2a^2)} \qquad (1)$$

be the equation to a tangent to the ellipse (Art. 171); then the equation to the perpendicular on it from the focus H is (see Fig. to Art. 175)

$$y = -\frac{1}{m}(x - ae) \qquad (2).$$

If we suppose x and y to have respectively the same values in (1) and (2), and eliminate m between the two equations, we shall obtain the required locus.

From (1) $\qquad y - mx = \sqrt{(b^2 + m^2a^2)}$;

from (2) $\qquad my + x = ae$;

square and add, then

$$(y^2 + x^2)(1 + m^2) = b^2 + m^2a^2 + a^2e^2$$
$$= a^2(1 + m^2);$$
$$\therefore\ y^2 + x^2 = a^2$$

is the equation to the required locus, which is therefore a circle described on the major axis of the ellipse as diameter.

We have supposed the perpendicular drawn from H; we shall arrive at the same result if it be drawn from S; hence if HZ, SZ' be these perpendiculars, CZ and CZ' each $= a$.

181. *To find the length of the perpendicular from the focus on the tangent at any point.*

The equation to the tangent at the point (x', y') is

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'}.$$

The co-ordinates of the focus H are ae, 0. But if p denote the length of the perpendicular from a point (x_1, y_1) on the line $y = mx + c$, by Art. 47

$$p^2 = \frac{(y_1 - mx_1 - c)^2}{1 + m^2}.$$

In the present case

$$x_1 = ae, \qquad y_1 = 0,$$
$$m = -\frac{b^2x'}{a^2y'}, \qquad c = \frac{b^2}{y'};$$

mult. by $(-1)^2$.

$$\therefore\ p^2 = \frac{\left(\dfrac{b^2x'ae}{a^2y'} - \dfrac{b^2}{y'}\right)^2}{1 + \dfrac{b^4x'^2}{a^4y'^2}} = \frac{a^2b^4(a - ex')^2}{a^4y'^2 + b^4x'^2}$$

$$= \frac{a^2b^4(a - ex')^2}{a^2(a^2b^2 - b^2x'^2) + b^4x'^2} = \frac{a^2b^2(a - ex')^2}{a^2(a^2 - e^2x'^2)}$$

$$= \frac{b^2(a - ex')}{a + ex'} = \frac{b^2r}{r'}, \quad \text{(Art. 177).}$$

Since $r' = 2a - r$ we have $p^2 = \dfrac{b^2r}{2a - r}$.

Similarly if p' be the perpendicular from S on the tangent at (x', y') we shall find

$$p'^2 = \frac{b^2r'}{r};$$

$$\therefore\ pp' = b^2.$$

182. *From any external point two tangents can be drawn to an ellipse.*

Let the equation to the ellipse be

$$a^2y^2 + b^2x^2 = a^2b^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and let h, k be the co-ordinates of an external point. Suppose x', y' the co-ordinates of a point on the ellipse, such that the tangent at this point passes through (h, k). The equation to the tangent at (x', y') is

$$a^2yy' + b^2xx' = a^2b^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Since this tangent passes through (h, k)

$$a^2ky' + b^2hx' = a^2b^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Also since (x', y') is on the ellipse

$$a^2y'^2 + b^2x'^2 = a^2b^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Equations (3) and (4) determine the values of x' and y'.

Substitute from (3) in (4), thus

$$\left(\frac{a^2b^2 - b^2hx'}{ak}\right)^2 + b^2x'^2 = a^2b^2,$$

or $$x'^2(a^2k^2 + b^2h^2) - 2a^2b^2hx' + a^4(b^2 - k^2) = 0.$$

The roots of this quadratic will be found to be both possible since (h, k) is an *external* point and therefore $a^2k^2 + b^2h^2$ greater than a^2b^2.

The line which passes through the points where these tangents meet the ellipse is called the *chord of contact.*

183. *Tangents are drawn to an ellipse from a given external point; to find the equation to the chord of contact.*

Let h, k be the co-ordinates of the external point; x_1, y_1 the co-ordinates of the point where one of the tangents from (h, k) meets the ellipse; x_2, y_2 the co-ordinates of the point where the other tangent from (h, k) meets the ellipse.

The equation to the tangent at (x_1, y_1) is

$$a^2yy_1 + b^2xx_1 = a^2b^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

since this tangent passes through (h, k) we have

$$a^2ky_1 + b^2hx_1 = a^2b^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Similarly, since the tangent at (x_2, y_2) passes through (h, k)

$$a^2ky_2 + b^2hx_2 = a^2b^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Hence it follows that the equation to the *chord of contact* is

$$a^2ky + b^2hx = a^2b^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

For (4) is obviously the equation to *some straight line* also this line passes through (x_1, y_1) for (4) is satisfied by the values $x = x_1$, $y = y_1$ as we see from (2); similarly from (3) we conclude that this line passes through (x_2, y_2). Hence (4) is the required equation.

Thus we may proceed as follows in order to draw tangents to an ellipse from a given external point—draw the line

which is represented by (4); join the points where it meets the ellipse with the given external point, and the lines thus obtained are the required tangents.

184. *Through any fixed point chords are drawn to an ellipse, and tangents to the ellipse are drawn at the extremities of each chord; the locus of the intersection of the tangents is a straight line.*

Let h, k be the co-ordinates of the point through which the chords are drawn; let tangents to the ellipse be drawn at the extremities of one of these chords, and let (x_1, y_1) be the point in which they meet. The equation to the corresponding chord of contact is, by Art. 183,

$$a^2yy_1 + b^2xx_1 = a^2b^2.$$

But this chord passes through (h, k); therefore

$$a^2ky_1 + b^2hx_1 = a^2b^2.$$

Hence the point (x_1, y_1) lies on the line

$$a^2ky + b^2hx = a^2b^2;$$

that is, the locus of the intersection of the tangents is a straight line.

We will now prove the converse of this proposition.

185. *If from any point in a straight line a pair of tangents be drawn to an ellipse the chords of contact will all pass through a fixed point.*

Let $$Ax + By + C = 0 \quad\text{.....................(1)}$$

be the equation to the straight line; let (x', y') be a point in this line from which tangents are drawn to the ellipse; then the equation to the corresponding chord of contact is

$$a^2yy' + b^2xx' = a^2b^2 \quad\text{.....................(2).}$$

Since (x', y') is on (1)

$$Ax' + By' + C = 0;$$

therefore (2) may be written

$$b^2xx' - \frac{Ax'+C}{B}a^2y = a^2b^2,$$

or,
$$\left(b^2x - \frac{Aa^2y}{B}\right)x' - \frac{Ca^2y}{B} - a^2b^2 = 0 \quad\ldots\ldots\ldots\ldots(3).$$

Now, whatever be the value of x', this line passes through the point whose co-ordinates are found by the simultaneous equations

$$b^2x - \frac{Aa^2y}{B} = 0, \qquad \frac{Ca^2y}{B} + a^2b^2 = 0,$$

that is, the point for which

$$y = -\frac{Bb^2}{C}, \qquad x = -\frac{Aa^2}{C}.$$

186. The student should observe the different interpretations that can be assigned to the equation

$$a^2ky + b^2hx = a^2b^2.$$

The statements in Art. 103 with respect to the circle may all be applied to the ellipse.

Chap IX

EXAMPLES.

1. What is the excentricity of the ellipse $2x^2 + 3y^2 = c^2$?

2. Find the equation to the tangent at the end of the latus rectum L. (See Fig. to Art. 162.) Also find the lengths of the intercepts of this tangent on the axes.

3. Write down the equation to the normal at L.

4. If the normal at L passes through the extremity of the minor axis B', what is the excentricity of the ellipse?

5. Find the equations to $A'B$ and CL. (See Fig. to Art. 162.) What is the excentricity of the ellipse if these lines are parallel?

6. Find the equation to $B'H$, and determine the abscissa of the point where this line cuts the ellipse again.

7. Find the equation to AL, and determine the angle between this line and the tangent at L.

8. If from the point P whose abscissa is x', a line be drawn through H, determine the abscissa of the point where it meets the ellipse again.

9. Find a point in the ellipse such that the tangent there is equally inclined to the axes.

10. Find a point in the ellipse such that the intercepts made by the tangent on the co-ordinate axes are proportional to the corresponding axes of the ellipse.

11. P is a point on an ellipse, y its ordinate; shew that

$$\tan APA' = -\frac{2b^2}{ae^2y}.$$

12. P is a point on an ellipse, y its ordinate; shew that the tangent of the angle between the focal distance and the tangent at P is $\frac{b^2}{aey}$.

13. If ϕ denote the angle mentioned in the preceding question,

$$PC = \sqrt{(a^2 - b^2 \cot^2 \phi)}.$$

14. From P a point in an ellipse lines are draw to A, A', the extremities of the major axis, and from A, A' lines are drawn perpendicular to AP, $A'P$; shew that the locus of their intersection will be another ellipse, and find its axes.

15. If any ordinate MP be produced to meet the tangent at L in Q, prove that $QM = PH$. (See Fig. to Art. 162.)

16. If a series of ellipses be described having the same major axes the tangents at the ends of their latera recta will pass through one or other of two fixed points.

17. If the focus of an ellipse be the common focus of two parabolas whose vertices are at the ends of the axis major, these parabolas will intersect at right angles, at points whose distance from each other is equal to twice the minor axis.

18. Shew that the length of the longer normal drawn from a point in the minor axis of an ellipse at a distance c from the centre and intercepted between that point and the curve is

$$\left(a^2 + \frac{c^2}{e^2}\right)^{\frac{1}{2}}.$$

19. If any parallel straight lines be drawn from the focus H and the extremity A of the axis major of an ellipse, and if M and N be the points where they meet the axis minor, or the axis minor produced, then the circle whose centre is M and radius NA will either *touch* the ellipse, or fall entirely *outside* of it.

20. A and A' are the extremities of the major axis of an ellipse, T is the point where the tangent at the point P of the curve meets AA' produced; through T a line is drawn perpendicular to AA' and meeting AP and $A'P$ produced in Q and R respectively; shew that $QT = RT$.

21. If ϕ, ϕ' be the excentric angles of two points, the equation to the chord joining the points is

$$\frac{x}{a}\cos\frac{\phi+\phi'}{2} + \frac{y}{b}\sin\frac{\phi+\phi'}{2} = \cos\frac{\phi-\phi'}{2}.$$

22. Express the equation to the tangent at any point in terms of the excentric angle of that point.

23. Shew that the equation to the normal at the point whose excentric angle is ϕ is

$$ax\sec\phi - by\,\mathrm{cosec}\,\phi = a^2 - b^2.$$

24. The locus of the middle point of PG (see Art. 176) is an ellipse of which the excentricity e' is connected with that of the given ellipse by the equation

$$1 - e^2 = (1 + e'^2)^2 (1 - e'^2).$$

25. Determine the point of intersection of the tangent at L with the line HB; what is the value of the excentricity of the ellipse when these lines are parallel?

26. A tangent at any point P of an ellipse meets the directrix EK in T and $E'K'$ in T', shew that TE varies as the cotangent of PHS and $T'E'$ varies as the cotangent of PSH. (See Fig. to Art. 162.)

27. If the straight line $y = mx + c$ intersect the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, shew that the length of the chord will be

$$\frac{2ab\sqrt{\{(1+m^2)(m^2a^2+b^2-c^2)\}}}{m^2a^2+b^2}.$$

Hence find the relation between the constants that this line may be a tangent to the ellipse.

28. Find the equation to the circle described on HP as diameter, supposing x', y' the co-ordinates of P.

29. Shew that any circle described on HP as diameter, touches the circle described on the major axis as diameter.

30. From a point (h, k) two tangents are drawn to an ellipse; find the sum of the perpendiculars from the foci on the chord of contact.

31. Any ordinate PM of an ellipse is produced to meet the circle on the axis major in Q and normals to the ellipse and circle at P and Q respectively meet in R; find the locus of R.

32. Two ellipses have a common centre and their axes coincide in direction; also the sum of the squares of the axes is the same in the two ellipses; find the equation to a common tangent.

33. If θ, θ' be the inclinations to the major axis of the ellipse of the two tangents that can be drawn from the point (h, k), shew that

$$\tan\theta + \tan\theta' = -\frac{2hk}{a^2-h^2}, \quad \tan\theta\tan\theta' = \frac{b^2-k^2}{a^2-h^2}.$$

34. Find the locus of a point such that the two tangents from it to an ellipse are at right angles.

35. Shew that the two tangents which can be drawn to an ellipse through the point (h, k) are represented by

$$(a^2 - h^2)(y - k)^2 + 2(y - k)(x - h)hk + (b^2 - k^2)(x - h)^2 = 0,$$

or by

$$(a^2k^2 + b^2h^2 - a^2b^2)(a^2y^2 + b^2x^2 - a^2b^2) = (a^2ky + b^2hx - a^2b^2)^2.$$

36. Tangents are drawn to an ellipse from the point (h, k); shew that the lines drawn from the origin to the points of contact are represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{xh}{a^2} + \frac{ky}{b^2}\right)^2.$$

37. Pairs of radii vectores are drawn at right angles to each other from the centre of an ellipse; shew that the tangents at their extremities intersect in the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}.$$

38. From an external point T whose co-ordinates are h and k a line is drawn to the centre C meeting the ellipse in R, shew that

$$\frac{CT^2}{CR^2} = \frac{a^2k^2 + b^2h^2}{a^2b^2}.$$

39. From an external point (h, k) tangents are drawn; if x_1, x_2 be the abscissæ of the points of contact, shew that

$$x_1 + x_2 = \frac{2ha^2b^2}{a^2k^2 + b^2h^2}, \quad x_1x_2 = \frac{a^4(b^2 - k^2)}{a^2k^2 + b^2h^2}.$$

40. From an external point (h, k) tangents are drawn meeting the ellipse in P and Q; find the value of $HP.HQ$, H being a focus.

41. From an external point T the lines TP, TQ are drawn to touch the ellipse in P and Q. CT cuts the ellipse

in R, and RN is drawn parallel to HT to meet the axis major in N; shew that $HP.HQ = RN^2$.

42. Two ellipses of equal excentricity and whose major axes are parallel can only have two points in common. Prove this, and shew that if three such ellipses intersect, two and two in the points P and P', Q and Q', R and R', respectively, the lines PP', QQ', RR', meet in a point.

43. Two concentric ellipses which have their axes in the same direction intersect, and four common tangents are drawn so as to form a rhombus, and the points of intersection of the ellipses are joined so as to form a rectangle; prove that the product of the areas of the rhombus and rectangle is equal to half the continued product of the four axes.

44. If the ordinate at any point P of an ellipse be produced to meet the circle described on the major axis as diameter in Q, prove that the perpendicular from the focus S on the tangent at Q is equal to SP.

45. Find the equation to the ellipse referred to axes passing through the extremities of the minor axis, and meeting in one extremity of the major axis.

46. If from points of the curve $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$, tangents be drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the chords of contact will be normal to the ellipse.

47. Prove the proposition in Art. 180 in a manner similar to that used in Art. 138. Also prove the proposition in Art. 138 in a manner similar to that used in Art. 180.

48. Find the equation to the ellipse the origin being the point (h, k) on the ellipse and the axes parallel to the axes of the ellipse.

49. From a point P on an ellipse two chords PQ, PQ' are drawn meeting the ellipse in Q, Q'; if h, k be the co-ordinates of P referred to the centre, and $mx + ny = 1$ the equation

to QQ' referred to P as origin, shew that the lines PQ, PQ' are represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \left(\frac{2xh}{a^2} + \frac{2yk}{b^2}\right)(mx + ny) = 0$$

with P as origin.

50. Let P be any point on an ellipse; draw PP' parallel to the major axis and cutting the curve in P'; through P draw two chords PQ, PQ', making equal angles with the major axis; join QQ'; QQ' shall be parallel to the tangent at P'.

51. From the equation $y = mx + \sqrt{(m^2a^2 + b^2)}$ deduce the equation to the tangent to the parabola.

52. In the figure of Art. 175 suppose GP produced to a point Q such that $GQ = n \,.\, GP$, and find the locus of Q.

53. If PN be any ordinate of a circle, and from the extremity A of the corresponding diameter AB, AQ be drawn meeting PN in Q, so that $AQ = PN$, find the locus of Q and the position of its focus.

54. Express the tangent of the angle between CP and the normal at P in terms of the co-ordinates of P.

55. Find the greatest value of the tangent of the angle between CP and the normal at P.

56. The major axis of an ellipse is equal to twice the minor axis; a line of length equal to half the major axis is placed with one end on the curve and the other on the minor axis; shew that the middle point of the line is on the major axis.

57. A circle is inscribed in the triangle formed by two focal distances and the major axis of an ellipse; find the locus of the centre.

58. If SZ', HZ be perpendiculars on the tangent at the point P of an ellipse, SZ and HZ' will intersect on the normal at P.

CHAPTER X.

THE ELLIPSE CONTINUED.

Diameters.

187. *To find the length of a line drawn from any point in a given direction to meet an ellipse.*

Let x', y' be the co-ordinates of the point from which the line is drawn; x, y the co-ordinates of the point to which the line is drawn; θ the inclination of the line to the axis of x; r the length of the line; then (Art. 27)

$$x = x' + r\cos\theta, \qquad y = y' + r\sin\theta \;\dots\dots\dots\; (1).$$

If (x, y) be on the ellipse these values may be substituted in the equation

$$a^2y^2 + b^2x^2 = a^2b^2; \text{ thus}$$

$$a^2(y' + r\sin\theta)^2 + b^2(x' + r\cos\theta)^2 = a^2b^2;$$

$$\therefore r^2(a^2\sin^2\theta + b^2\cos^2\theta) + 2r(a^2y'\sin\theta + b^2x'\cos\theta)$$
$$+ a^2y'^2 + b^2x'^2 - a^2b^2 = 0 \;\dots\dots\dots\dots\dots\; (2).$$

From this quadratic two values of r can be found which are the lengths of the two lines that can be drawn from (x', y') in the given direction to the ellipse.

188. *To find the diameter of a given system of parallel chords in an ellipse.* (See definition in Art. 148.)

Let θ be the inclination of the chords to the major axis of the ellipse; let x', y' be the co-ordinates of the middle point of any one of the chords; the equation which determines the

lengths of the lines drawn from (x', y') to the curve is (Art. 187)

$$r^2 (a^2 \sin^2 \theta + b^2 \cos^2 \theta) + 2r (a^2 y' \sin \theta + b^2 x' \cos \theta)$$
$$+ a^2 y'^2 + b^2 x'^2 - a^2 b^2 = 0 \text{ .... } (1).$$

Since (x', y') is the *middle* point of the chord, the values of r furnished by this quadratic must be *equal in magnitude and opposite in sign;* hence the coefficient of r must vanish; thus

$$a^2 y' \sin \theta + b^2 x' \cos \theta = 0, \quad \text{or } y' = -\frac{b^2}{a^2} \cot \theta \, . \, x' \, . \qquad (2).$$

Considering x' and y' as variable, this is the equation to a straight line passing through the origin, that is, through the centre of the ellipse.

Hence every diameter passes through the centre.

Also every straight line passing through the centre is a diameter, that is, bisects some system of parallel chords; for by giving to θ a suitable value the equation (2) may be made to represent *any* line passing through the centre.

If θ' be the inclination to the axis of x of the diameter which bisects all the chords inclined at an angle θ we have from (2)

$$\tan \theta' = -\frac{b^2}{a^2} \cot \theta ;$$

$$\tan \theta \tan \theta' = -\frac{b^2}{a^2} \qquad (3).$$

189. *If one diameter bisect all chords parallel to a second diameter, the second diameter will bisect all chords parallel to the first.*

Let θ_1 and θ_2 be the respective inclinations of the two diameters to the major axis of the ellipse. Since the first bisects all the chords parallel to the second, we have

$$\tan \theta_2 \tan \theta_1 = -\frac{b^2}{a^2} .$$

And this is also the only condition that must hold in order that the second may bisect the chords parallel to the first.

190. *The tangent at either extremity of any diameter is parallel to the chords which that diameter bisects.*

Let h, k be the co-ordinates of either extremity of a diameter; θ the inclination to the major axis of the ellipse of the chords which the diameter bisects. Then the values $x=h$, $y=k$ must satisfy the equation

$$a^2y\sin\theta + b^2x\cos\theta = 0;$$

$$\therefore\ \tan\theta = -\frac{b^2h}{a^2k}.$$

But, by Art. 170, the equation to the tangent at (h, k) is

$$y-k = -\frac{b^2h}{a^2k}(x-h).$$

Hence the tangent is parallel to the bisected chords.

191. Def. Two diameters are called *conjugate* when each bisects the chords parallel to the other.

From Art. 190 it follows that each of the conjugate diameters is parallel to the tangent at either extremity of the other.

192. *Given the co-ordinates of one extremity of a diameter to find those of either extremity of the conjugate diameter.*

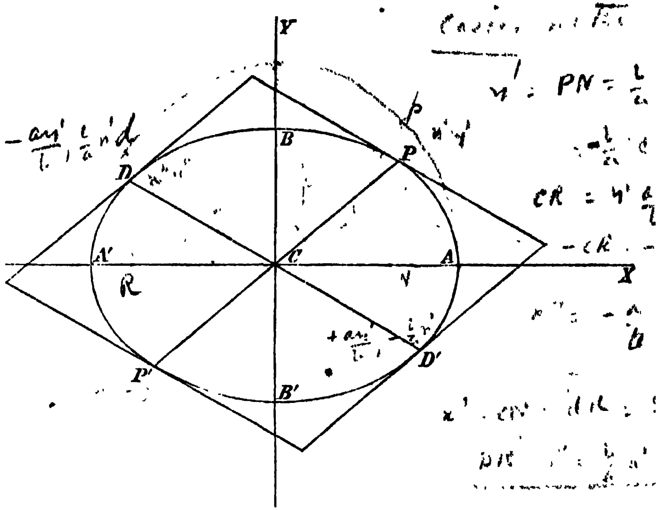
Let ACA', BCB' be the axes of an ellipse; PCP', DCD' a pair of conjugate diameters.

Let x', y' be the given co-ordinates of P; then the equation to CP is

$$y = \frac{y'}{x'}x. \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since the conjugate diameter DD' is parallel to the tangent at P, the equation to DD' is

$$y = -\frac{b^2x'}{a^2y'}x \quad\quad (2).$$



We must combine (2) with the equation to the ellipse to find the co-ordinates of D and D'. Substitute from (2) in

$$a^2y^2 + b^2x^2 = a^2b^2;$$

then
$$a^2\frac{b^4}{a^4}\frac{x'^2}{y'^2}x^2 + b^2x^2 = a^2b^2;$$

$$(b^2x'^2 + a^2y'^2)\,x^2 = a^4y'^2;$$

$$\therefore\ x^2 = \frac{a^4y'^2}{a^2b^2} = \frac{a^2y'^2}{b^2};$$

$$\therefore\ x = \pm\frac{ay'}{b};$$

$$\therefore \text{ from (2) } y = \mp\, bx'$$

In the figure the abscissa of D is negative and that of D' positive; hence the upper sign applies to D' and the lower to D.

The properties of the ellipse connected with conjugate diameters are numerous and important; we shall now give a few of them.

193. *The sum of the squares of two conjugate semi-diameters is constant.*

Let x', y' be the co-ordinates of P; then by the preceding article

$$CP^2 + CD^2 = x'^2 + y'^2 + \frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2}$$

$$= \frac{a^2y'^2 + b^2x'^2}{b^2} + \frac{a^2y'^2 + b^2x'^2}{a^2}$$

$$= a^2 + b^2.$$

Thus the sum of the squares of two conjugate semi-diameters is equal to the sum of the squares of the semi-axes.

Moreover

$$CD^2 = a^2 + b^2 - x'^2 - y'^2 = a^2 + b^2 - x'^2 - \frac{b^2}{a^2}(a^2 - x'^2)$$

$$= a^2 - \left(1 - \frac{b^2}{a^2}\right)x'^2 = a^2 - e^2x'^2 = SP \,.\, HP \text{ by Art. 166.}$$

194. *The area of the parallelogram which touches the ellipse at the ends of conjugate diameters is constant.*

Let PCP', DCD' be the conjugate diameters (see Fig. to Art. 192). The area of the parallelogram described so as to touch the ellipse at P, D, P', D', is $4CP \,.\, CD \sin PCD$, or $4p \,.\, CD$, where p denotes the perpendicular from C on the tangent at P. Let x', y' be the co-ordinates of P; then the equation to the tangent at P is

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'}.$$

Hence (Art. 47) $$p = \frac{\frac{b^2}{y'}}{\sqrt{\left(1 + \frac{b^4x'^2}{a^4y'^2}\right)}} = \frac{a^2b^2}{\sqrt{(a^4y'^2 + b^4x'^2)}}.$$

And $$CD = \sqrt{\left(\frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2}\right)} = \frac{\sqrt{(a^4y'^2 + b^4x'^2)}}{ab};$$

$$\therefore\ 4p \,.\, CD = 4ab.$$

Thus the area of any parallelogram which touches the ellipse at the ends of conjugate diameters is equal to the area

of the rectangle which touches the ellipse at the ends of the axes.

195. Let a', b' denote the lengths of two conjugate semi-diameters; α the angle between them; by the preceding article

$$a'b' \sin \alpha = ab;$$

$$\therefore \sin^2 \alpha = \frac{a^2b^2}{a'^2b'^2} = \frac{4a^2b^2}{(a'^2+b'^2)^2 - (a'^2-b'^2)^2} = \frac{4a^2b^2}{(a^2+b^2)^2 - (a'^2-b'^2)^2}.$$

Hence $\sin^2 \alpha$ has its *least* value when $a' = b'$, and then

$$\sin \alpha = \frac{2ab}{a^2+b^2}.$$

196. From Art. 194 we have

$$p^2 = \frac{a^2b^2}{CD^2} = \frac{a^2b^2}{a^2+b^2-CP^2} \quad \text{(Art. 193).}$$

This gives a relation between p the perpendicular from the centre on the tangent at any point P and the distance CP of that point from the centre.

We may also express p in terms of the angle its direction makes with the axis major; for let ψ denote the angle, then the equation to the tangent at (x', y') may be written

$$a^2yy' + b^2xx' = a^2b^2,$$

and also in the form (Art. 20)

$$x \cos \psi + y \sin \psi = p.$$

Hence
$$\frac{p}{\sin \psi} = \frac{b^2}{y'}, \qquad \frac{p}{\cos \psi} = \frac{a^2}{x'};$$

$$\therefore ay' = \frac{ab^2 \sin \psi}{p},$$

$$bx' = \frac{a^2b \cos \psi}{p};$$

$$\text{and } \therefore a^2b^2 = \frac{a^2b^2}{p^2}(b^2 \sin^2 \psi + a^2 \cos^2 \psi);$$

$$\therefore p^2 = b^2 \sin^2 \psi + a^2 \cos^2 \psi$$
$$= a^2(1 - e^2 \sin^2 \psi).$$

197. Let ϕ and ϕ' be the excentric angles corresponding to P and D respectively (Art. 168). Then

$$x' = a\cos\phi \ldots\ldots\ldots(1), \qquad y' = b\sin\phi \ldots\ldots\ldots(2),$$

$$\frac{ay'}{b} = a\cos\phi' \ldots\ldots\ldots(3), \qquad \frac{bx'}{a} = b\sin\phi'. \ldots\ldots\ldots(4).$$

From (2) and (3) $\cos\phi' = -\sin\phi$,

from (1) and (4) $\sin\phi' = \cos\phi$;

$$\therefore\ \phi' = \frac{\pi}{2} + \phi.$$

198. *To find the equation to the ellipse referred to a pair of conjugate diameters as axes.*

Let CP, CD be two conjugate semi-diameters (see fig. to Art. 192), take CP as the new axis of x, CD as that of y; let $PCA = \alpha$, $DCA = \beta$. Let x, y be the co-ordinates of any point of the ellipse referred to the original axes; x', y' the co-ordinates of the same point referred to the new axes; then (Art. 84)

$$\left.\begin{aligned} x &= x'\cos\alpha + y'\cos\beta, \\ y &= x'\sin\alpha + y'\sin\beta. \end{aligned}\right\}$$

Substitute these values in the equation

$$a^2y^2 + b^2x^2 = a^2b^2;$$

then $a^2(x'\sin\alpha + y'\sin\beta)^2 + b^2(x'\cos\alpha + y'\cos\beta)^2 = a^2b^2$,

or $x'^2(a^2\sin^2\alpha + b^2\cos^2\alpha) + y'^2(a^2\sin^2\beta + b^2\cos^2\beta)$

$$+ 2x'y'(a^2\sin\alpha\sin\beta + b^2\cos\alpha\cos\beta) = a^2b^2.$$

But, since CP and CD are conjugate semi-diameters,

$$\tan\alpha\tan\beta = -\frac{b^2}{a^2};$$

hence the coefficient of $x'y'$ vanishes, and the equation becomes

$$x'^2(a^2\sin^2\alpha + b^2\cos^2\alpha) + y'^2(a^2\sin^2\beta + b^2\cos^2\beta) = a^2b^2.$$

In this equation, suppose $x' = 0$, then

$$y'^2 = \frac{a^2b^2}{a^2 \sin^2 \beta + b^2 \cos^2 \beta}.$$

This is the value of CD^2, which we shall denote by b'^2; similarly we shall denote CP^2 by a'^2, so that

$$= \frac{a^2b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}.$$

Hence the equation to the ellipse referred to conjugate diameters is

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1,$$

or, suppressing the accents on the variables,

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$$

199. A particular case of the preceding is when $a' = b'$; then

$$a^2 \sin^2 \beta + b^2 \cos^2 \beta = a^2 \sin^2 \alpha + b^2 \cos^2 \alpha;$$

$$\therefore\ a^2 (\sin^2 \beta - \sin^2 \alpha) = b^2 (\cos^2 \alpha - \cos^2 \beta)$$

$$= b^2 (\sin^2 \beta - \sin^2 \alpha);$$

$$\therefore\ (a^2 - b^2)(\sin^2 \beta - \sin^2 \alpha) = 0;$$

$$\therefore\ \sin^2 \beta = \sin^2 \alpha;$$

$$\therefore\ \beta = \pi - \alpha.$$

And since $a'^2 = b'^2$ each of them $= \dfrac{a^2 + b^2}{2}$, (Art. 193).

Hence from the value of a'^2 in the preceding article, we have

$$\frac{a^2 + b^2}{2} = \frac{a^2b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha},$$

$$\therefore\ (a^2 + b^2)\{(a^2 - b^2) \sin^2 \alpha + b^2\} = 2a^2b^2;$$

$$\therefore\ \sin^2 \alpha = \frac{a^2b^2 - b^4}{(a^2 + b^2)(a^2 - b^2)} = \frac{b^2}{a^2 + b^2}$$

This shews that the *equal* conjugate diameters are parallel to the lines BA and BA'.

200. The equation to the tangent to the ellipse will be of *the same form* whether the axes be rectangular or the oblique system formed by a pair of conjugate diameters; for the investigation of Art. 170 will apply without any change to the equation $a'^2y^2 + b'^2x^2 = a'^2b'^2$ which represents an ellipse referred to such an oblique system.

201. *Tangents at the extremities of any chord of an ellipse meet in the diameter which bisects that chord.*

Refer the ellipse to the diameter bisecting the chord as the axis of x, and the diameter parallel to the chord as the axis of y; let the equation to the ellipse be

$$a'^2y^2 + b'^2x^2 = a'^2b'^2.$$

Let x', y' be the co-ordinates of one extremity of the chord; then the equation to the tangent at this point is

$$a'^2yy' + b'^2xx' = a'^2b'^2 \quad\text{.................. (1).}$$

The co-ordinates of the other extremity of the chord are x', $-y'$, and the equation to the tangent there is

$$-a'^2yy' + b'^2xx' = a'^2b'^2 \quad\text{.................. (2).}$$

The lines represented by (1) and (2) meet at the point for which

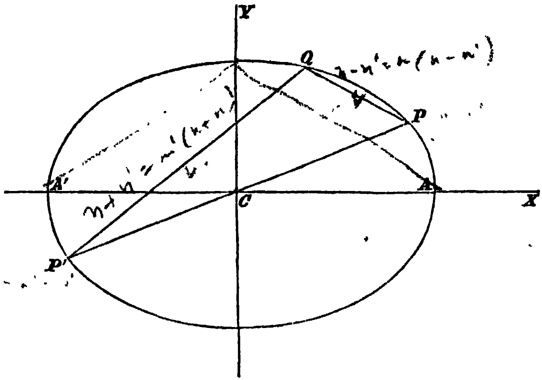
$$y = 0, \qquad x = \frac{a'^2}{x'};$$

this proves the proposition.

Supplemental chords.

202. Def. Two straight lines drawn from a point of the ellipse to the extremities of any diameter are called *supplemental* chords. They are called *principal* supplemental chords if that diameter be the major axis.

203. *If a chord and diameter of an ellipse are parallel, the supplemental chord is parallel to the conjugate diameter.*



Let PP' be a diameter of the ellipse; QP, QP' two supplemental chords. Let x', y' be the co-ordinates of P, and therefore $-x'$, $-y'$ the co-ordinates of P'.

Let the equation to PQ be (Art. 32)

$$y - y' = m(x - x') \quad \text{..............(1)},$$

and the equation to $P'Q$

$$y + y' = m'(x + x') \quad \text{..................(2)}.$$

The co-ordinates of the point Q satisfy (1) and (2); if then we suppose x, y to denote those co-ordinates, we have from (1) and (2) by multiplication

$$y^2 - y'^2 = mm'(x^2 - x'^2) \quad \text{..................(3)}.$$

But since (x, y) and (x', y') are points on the ellipse

$$a^2y^2 + b^2x^2 = a^2b^2,$$

$$a^2y'^2 + b^2x'^2 = a^2b^2;$$

$$\therefore\ a^2(y^2 - y'^2) + b^2(x^2 - x'^2) = 0;$$

$$\therefore\ y^2 - y'^2 = -\frac{b^2}{a^2}(x^2 - x'^2) \quad \text{..................(4)};$$

From (3) and (4) we have

$$mm' = -\frac{b^2}{a^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

But we have shewn in Art. 188 that if (5) be satisfied, the two lines represented by $y = mx$ and $y = m'x$ are conjugate diameters; this proves the theorem.

Polar Equation.

204. To find the polar equation to the ellipse, the focus being the pole.

Let $SP = r$, $A'SP = \theta$, (see Fig. to Art. 158);

then $\qquad SP = ePN$, by definition;

that is, $\qquad SP = e\,(OS + SM)$;

or $\qquad r = a\,(1 - e^2) + er\cos(\pi - \theta)$, (Art. 161);

$$\therefore\ r\,(1 + e\cos\theta) = a\,(1 - e^2),$$

and $$r = \frac{a\,(1 - e^2)}{1 + e\cos\theta}.$$

If we denote the angle ASP by θ, then we have as before

$$SP = e\,(OS + SM);$$

thus $$r = a\,(1 - e^2) + er\cos\theta,$$

and $$r = \frac{a\,(1 - e^2)}{1 - e\cos\theta}.$$

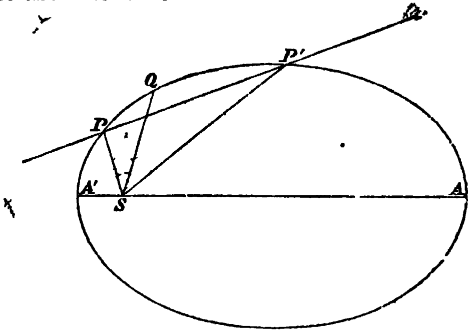
205. We shall make use of the preceding article in finding the polar equation to a chord, from which we shall deduce the polar equation to the tangent.

Let P and P' be two points on the ellipse; suppose

$$A'SP = \alpha - \beta,\quad A'SP' = \alpha + \beta,$$

so that $PSP' = 2\beta$; and let l be the semi-latus rectum of the

ellipse, so that $l = a(1-e^2)$; it is required to find the polar equation to the line PP'.



Assume for the equation (see Art. 29)

$$Ar\cos\theta + Br\sin\theta + C = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Since the line passes through P, equation (1) must be satisfied by the co-ordinates of P; now $A'SP = \alpha - \beta$, and therefore $SP = \dfrac{l}{1 + e\cos(\alpha-\beta)}$; thus from (1)

$$l\{A\cos(\alpha-\beta) + B\sin(\alpha-\beta)\}$$
$$+ C\{1 + e\cos(\alpha-\beta)\} = 0 \ldots\ldots\ldots\ldots\ldots(2).$$

Similarly, since the line passes through P',

$$l\{A\cos(\alpha+\beta) + B\sin(\alpha+\beta)\}$$
$$+ C\{1 + e\cos(\alpha+\beta)\} = 0 \ldots\ldots\ldots\ldots(3).$$

From (2) and (3), by subtraction,

$$l(A\sin\alpha\sin\beta - B\cos\alpha\sin\beta) + Ce\sin\alpha\sin\beta = 0;$$

$$\therefore\ l(A\sin\alpha - B\cos\alpha) + Ce\sin\alpha = 0 \ldots\ldots\ldots\ldots(4).$$

From (2) and (3), by addition,

$$l(A\cos\alpha\cos\beta + B\sin\alpha\cos\beta) + C(1 + e\cos\alpha\cos\beta) = 0;$$

$$\therefore\ l(A\cos\alpha + B\sin\alpha) + C(\sec\beta + e\cos\alpha) = 0 \ldots\ldots\ldots(5).$$

From (4) and (5) we find

$$lA + C(\sec\beta\cos\alpha + e) = 0,$$

$$lB + C\sec\beta\sin\alpha \qquad = 0.$$

Substitute the values of A and B in (1) and divide by C, and we have

$$r\{(\sec\beta\cos\alpha + e)\cos\theta + \sec\beta\sin\alpha\sin\theta\} - l = 0;$$

$$\therefore\ r = \frac{l}{e\cos\theta + \sec\beta\cos\cancel{(\alpha - \theta)}}.$$

If SQ bisect the angle PSP', we have

$$PSQ = \beta, \text{ and } A'SQ = \alpha.$$

Now suppose β to diminish indefinitely; then the chord *PP' becomes the tangent at Q*, and we obtain its polar equation by putting $\beta = 0$ in the preceding result; thus we have

$$r = \frac{l}{e\cos\theta + \cos\cancel{(\alpha - \theta)}}.$$

The investigations of this article will apply to the parabola by supposing $e = 1$.

206. The polar equation to the ellipse referred to the centre is sometimes useful; it may be deduced from the equation $a^2y^2 + b^2x^2 = a^2b^2$, by putting $r\cos\theta$, $r\sin\theta$, for x and y respectively; we thus obtain

$$r^2(a^2\sin^2\theta + b^2\cos^2\theta) = a^2b^2.$$

We add a few miscellaneous propositions on the ellipse.

207. *If tangents be drawn at the extremities of any focal chord of an ellipse, (1) the tangents will intersect in the corresponding directrix, (2) the line drawn from the point of intersection of the tangents to the focus will be perpendicular to the focal chord.*

(1) If two tangents to an ellipse meet in the point (h, k) the equation to the chord of contact is, by Art. 183,

$$a^2ky + b^2hx = a^2b^2.$$

Suppose the chord passes through the focus whose co-ordinates are $x = -ae$, $y = 0$; then

$$-b^2hae = a^2b^2,$$

$$\therefore\ h = -\frac{a}{e};$$

that is, the point of intersection of the tangents is on the directrix corresponding to this focus.

(2) The equation to the line through (h, k) and the focus is

$$y = \frac{k}{h + ae}(x + ae).$$

If $h = -\dfrac{a}{e}$, this becomes

$$y = -\frac{ke}{a(1 - e^2)}(x + ae)$$

$$= \frac{ka^2}{hb^2}(x + ae),$$

and the line is therefore perpendicular to the focal chord of which the equation is

$$y = -\frac{b^2hx}{a^2k} + \frac{b^2}{k}.$$

208. *If through any point within or without an ellipse, two lines be drawn parallel to two given straight lines to meet the curve, the rectangles of the segments will be to one another in an invariable ratio.*

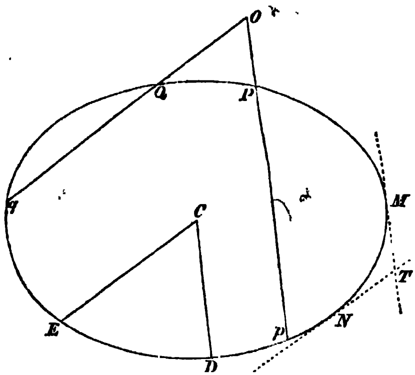
Let (x', y') be the given point and suppose α and β respectively the inclinations of the given straight lines to the major axis of the ellipse. By Art. 187 if a line be drawn from (x', y') to meet the curve and be inclined at an angle α to the major axis, the lengths of its segments are given by the equation

$$r^2 (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) + 2r (a^2 y' \sin \alpha + b^2 x' \cos \alpha) + a^2 y'^2 + b^2 x'^2 - a^2 b^2 = 0;$$

therefore the rectangle of the segments $= \dfrac{a^2 y'^2 + b^2 x'^2 - a^2 b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}$.

Similarly the rectangle of the segments of the line drawn from (x', y') at an angle $\beta = \dfrac{a^2 y'^2 + b^2 x'^2 - a^2 b^2}{a^2 \sin^2 \beta + b^2 \cos^2 \beta}$.

Hence the ratio of the rectangles $= \dfrac{a^2 \sin^2 \beta + b^2 \cos^2 \beta}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}$; and this ratio is constant whatever x' and y' may be.



Let O be the point through which the lines OPp, OQq, are drawn inclined to the major axis of the ellipse at angles α, β, respectively; then

$$\frac{OP \cdot Op}{OQ \cdot Oq} = \frac{a^2 \sin^2 \beta + b^2 \cos^2 \beta}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}.$$

Draw the semi-diameters CD, CE, parallel to Pp, Qq, respectively, then, by Art. 206,

$$\frac{CD^2}{CE^2} = \frac{a^2 \sin^2 \beta + b^2 \cos^2 \beta}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha};$$

$$\frac{OP \cdot Op}{OQ \cdot Oq} = \frac{CD^2}{CE^2}.$$

Let TM, TN be tangents parallel to Pp, Qq, respectively; then if O coincides with T, the rectangle $OP \cdot Op$ becomes TM^2, and the rectangle $OQ \cdot Oq$ becomes TN^2;

$$\therefore \frac{TM^2}{TN^2} = \frac{CD^2}{CE^2};$$

$$\therefore \frac{TM}{TN} = \frac{CD}{CE}.$$

Chap X

EXAMPLES.

1. CP and CD are conjugate semi-diameters; given the co-ordinates of P (x', y'), find the equation to the line PD.

2. If lines drawn through any point of an ellipse to the extremities of any diameter meet the conjugate CD in the points M, N, prove that $CM \cdot CN = CD^2$.

3. CP, CD are two conjugate semi-diameters; CP', CD' are two other conjugate diameters; shew that the area of the triangle PCP' is equal to the area of the triangle DCD'.

4. Normals at P and D, the extremities of semi-conjugate diameters, meet in K; find the equation to KC, and shew that KC is perpendicular to PD.

5. In an ellipse the rectangle contained by the perpendicular from the centre upon the tangent, and the part of the corresponding normal intercepted between the axes is equal to the difference of the squares of the semi-axes

6. Shew that the locus of the intersection of the perpendicular from the centre on a tangent to the ellipse is the curve which has for its equation $r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$, the centre being the origin.

7. From A the vertex of an ellipse draw a line ARQ to Q the middle point of HP meeting SP in R; shew that the locus of R is an ellipse, and also the locus of Q.

8. Find the polar equation to the ellipse, the vertex being the origin and the major axis the initial line.

9. If any chord AQ meet the minor axis produced in R, and CP be a semi-diameter parallel to AQ, then

$$AQ \, . \, AR = 2CP^2.$$

10. A circle is described upon AA' the major axis of an ellipse as diameter; P is any point in the circle; AP, $A'P$ are joined cutting the ellipse in points Q and Q' respectively; shew that

$$\frac{AP}{AQ} + \frac{A'P}{A'Q'} = \frac{a^2 + b^2}{b^2}.$$

11. If circles be described on two semi-conjugate diameters of an ellipse as diameters, the locus of their intersection is the curve defined by the equation

$$2(x^2 + y^2)^2 = a^2x^2 + b^2y^2.$$

12. CP, CD are conjugate semi-diameters; CQ is perpendicular to PD, find the locus of Q.

13. Find the points where the ellipse $a(1-e^2) = r + re\cos\theta$ cuts the line $a(1-e^2) = r\sin\theta + r(1+e)\cos\theta$.

14. Write down the polar equations to the four tangents at the ends of the latera recta; also the equations to the tangents at the ends of the minor axis; the focus being the pole.

15. Determine the locus of the intersection of tangents drawn at two points P, Q, which are taken so that the sum of the angles ASP, ASQ, is constant.

16. If PSp be a focal chord of an ellipse, and along the line SP there be set off SQ a mean proportional between SP and Sp, the locus of Q will be an ellipse having the same excentricity as the original ellipse.

17. Two ellipses have a common focus and their major axes are equal in length and situated in the same straight line; find the polar co-ordinates of the points of intersection.

18. From an external point two tangents are drawn to an ellipse; between what limits does the ratio of the length of one tangent to the other lie?

19. TP, TQ are two tangents to an ellipse, and CP', CQ', are the radii from the centre respectively parallel to these tangents, prove that $P'Q'$ is parallel to PQ.

20. An ellipse and a circle cut in four points; shew that the common chords make equal angles with the major axis of the ellipse.

21. When the angle between the radius vector from the *focus* and the tangent is least, the radius vector $= a$.

22. When the angle between the radius vector from the *centre* and the tangent is least, the radius vector $= \left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}$.

23. PT, pt are tangents at the extremities of any diameter Pp of an ellipse; any other diameter meets PT in T, and its conjugate meets pt in t; also any tangent meets PT in T' and pt in t'; shew that $PT : PT' :: pt' : pt$.

24. From the ends P, D, of conjugate diameters in an ellipse, draw lines parallel to any tangent line; and from the centre C draw any line cutting these lines and the tangent in points p, d, t, respectively; then will

$$Cp^2 + Cd^2 = Ct^2.$$

25. If tangents be drawn from different points of an ellipse of lengths equal to n times the semi-conjugate diameter at each point, then the locus of their extremities will be a concentric ellipse with semi-axes equal to

$$a\sqrt{(n^2+1)} \text{ and } b\sqrt{(n^2+1)}.$$

26. Apply the equation to the tangent in Art. 171 to find the locus of the intersection of tangents at the extremities of conjugate diameters.

27. If from a point (x', y') of an ellipse a chord be drawn parallel to a fixed line, shew that the length of this chord

varies as $\frac{y' \sin(\alpha - \phi)}{\cos \phi}$, where ϕ is the inclination of the tangent at (x', y') to the axis, and α the inclination of the fixed line to the axis.

28. If through any point P of an ellipse two chords PQ, PR, be drawn parallel to two fixed lines and making angles α and β respectively with the tangent at P, shew that the ratio PQ cosec α : PR cosec β is constant.

29. A parabola is touched at the extremities of the latus rectum by an ellipse of given magnitude; find the latus rectum of the parabola.

30. The perpendicular from the centre on a line joining the ends of perpendicular diameters of an ellipse is of constant length.

31. Chords are drawn through the end of an axis of an ellipse; find the locus of their middle points.

32. Chords of an ellipse are drawn through any fixed point; find the locus of their middle points.

33. Two focal chords are drawn in an ellipse at right angles to each other; find their position when the rectangle contained by them has respectively its greatest and least value.

34. In an ellipse if PP' and QQ' be focal chords at right angles to each other

$$\frac{1-e^2}{SP \cdot SP'} + \frac{1-e^2}{SQ \cdot SQ'} = \frac{1}{AC^2} + \frac{1}{BC^2}.$$

35. PSp, QSq, are focal chords; suppose T the point where the lines PQ, pq meet; shew that TS is equally inclined to the focal chords, and that T is on the directrix corresponding to S.

36. If r, θ be the polar co-ordinates of a point P, shew that

$$\tan HPZ = \frac{b}{\sqrt{(2ar - r^2 - b^2)}} \text{ and } = \frac{1 + e \cos \theta}{e \sin \theta}.$$

37. Perpendiculars are drawn from P and D the extremities of any pair of conjugate diameters on the diameter $y = x \tan \alpha$; shew that the sum of the squares of the perpendiculars is $a^2 \sin^2 \alpha + b^2 \cos^2 \alpha$.

38. The excentric angles of two points P and Q are ϕ and ϕ' respectively; shew that the area of the parallelogram formed by the tangents at the extremities of the diameters through P and Q is $\dfrac{4ab}{\sin(\phi' - \phi)}$; shew also that the area is least when P and Q are the extremities of conjugate diameters.

39. Shew that the equation to the locus of the middle points of all chords of the same length $(2c)$ in an ellipse is

$$c^2 \frac{a^2y^2 + b^2x^2}{a^4y^2 + b^4x^2} + \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

40. Chords of an ellipse are drawn at right angles to one another through a point O whose co-ordinates are h, k; if CP, CQ be the radii drawn from the centre parallel to the chords, and E, F the middle points of the chords, shew that

$$\frac{OE^2}{CP^4} + \frac{OF^2}{CQ^4} = \frac{h^2}{a^4} + \frac{k^2}{b^4}.$$

41. Given the co-ordinates of P, find those of the intersection of the tangents at P and D. (See Fig. to Art. 192.)

42. Shew that the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = \left\{\frac{x(bx' - ay')}{a^2b} + \frac{y(ay' + bx')}{ab^2} - 1\right\}^2$$

represents the tangents at P and D, supposing x', y' the co-ordinates of P. (See Fig. to Art. 192.)

43. If CP, CD be any conjugate diameters of an ellipse $APBDA'$, and BP, BD be joined and also AD, $A'P$, these latter intersecting in O, shew that $BDOP$ is a parallelogram.

44. Shew that the area of the parallelogram in the preceding question $=ay'+bx'-ab$, where x', y' are the co-ordinates of P; and find the greatest value of this area.

45. If a line be drawn from the focus of an ellipse to make a given angle α with the tangent, shew that the locus of its intersection with the tangent will be a circle which touches or falls entirely without the ellipse according as $\cos\alpha$ is less or greater than the excentricity of the ellipse.

46. In an ellipse SQ, HQ, drawn perpendicular to a pair of conjugate diameters, intersect in Q; prove that the locus of Q is a concentric ellipse.

47. Two ellipses have their foci coincident; a tangent to one of them intersects at right angles a tangent to the other; shew that the locus of the point of intersection is a circle having the same centre as the ellipses.

48. What is represented by the equation $x^2+y^2=c^2$ when the axes are oblique?

49. Shew that when the ellipse is referred to any pair of conjugate diameters as axes, the condition that $y=mx$ and $y=m'x$ may represent conjugate diameters is $mm'=-\dfrac{b'^2}{a'^2}$.

50. The ellipse being referred to equal conjugate diameters, find the equation to the normal at any point.

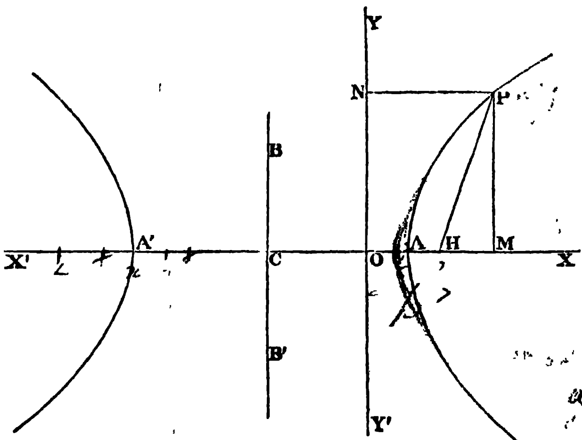
51. From any point P perpendiculars PM, PN are drawn on the equal conjugate diameters; shew that the normal at P bisects MN.

CHAPTER XI.

THE HYPERBOLA.

209. *To find the equation to the hyperbola.*

The hyperbola is the locus of a point which moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line, the ratio being greater than unity.



Let H be the fixed point, YY' the fixed straight line. Draw HO perpendicular to YY'; take O as the origin, OH as the direction of the axis of x, OY as that of the axis of y.

Let P be a point on the locus; join HP, draw PM parallel to OY and PN parallel to OX. Let $OH = p$, and let e be the ratio of HP to PN. Let x, y be the co-ordinates of P.

By definition

$$HP = ePN;$$

$$\therefore\ HP^2 = e^2PN^2;$$

$$\therefore\ PM^2 + HM^2 = e^2PN^2,$$

that is, $$y^2 + (x-p)^2 = e^2x^2.$$

This is the equation to the hyperbola with the assumed origin and axes.

210. To find where the hyperbola meets the axis of x we put $y = 0$ in the equation to the hyperbola; thus

$$(x-p)^2 = e$$

$$\therefore\ x - p = \pm ex;$$

$$\therefore\ x = \frac{p}{1 \mp e}.$$

Since e is greater than unity, $1 - e$ is a negative quantity.

Let $OA' = \frac{p}{e-1}$, $OA = \frac{p}{1+e}$, the former being measured to the *left* of O, then A' and A are points on the hyperbola. A and A' are called the *vertices* of the hyperbola, and C the point midway between A and A' is called the *centre* of the hyperbola.

211. We shall obtain a simpler form of the equation to the hyperbola by transferring the origin to A or C.

I. Suppose the origin at A.

Since $OA = \frac{p}{1+e}$, we put $x = x' + \frac{p}{1+e}$ and substitute this value in the equation

$$y^2 + (x-p)^2 = e^2x^2;$$

thus $$y^2 + \left(x' + \frac{p}{1+e} - p\right)^2 = e^2\left(x' + \frac{p}{1+e}\right)^2,$$

$$y^2 + \left(x' - \frac{ep}{1+e}\right)^2 = e^2\left(x' + \frac{p}{1+e}\right)^2;$$

$$\therefore\ y^2 + x'^2 - \frac{2x'ep}{1+e} = e^2\left(x'^2 + \frac{2px'}{1+e}\right);$$

$$\therefore\ y^2 = 2pex' + (e^2-1)\,x'^2$$

$$= (e^2-1)\left\{\frac{2pex'}{e^2-1} + x'^2\right\}.$$

The distance $A'A = \frac{p}{e-1} + \frac{p}{1+e} = \frac{2ep}{e^2-1}$; we will denote this by $2a$; hence the equation becomes $\therefore a = \frac{ep}{e^2-1}$

$$y^2 = (e^2-1)(2ax' + x'^2).$$

We may suppress the accent, if we remember that the origin is at the vertex A, and thus write the equation

$$y^2 = (e^2-1)(2ax + x^2) \text{ .... } (1).$$

II. Suppose the origin at C.

Since $CA = a$, we put $x = x' - a$ and substitute this value in (1); thus

$$y^2 = (e^2-1)\{2a(x'-a) + (x'-a)^2\}$$

$$= (e^2-1)(x'^2 - a^2).$$

We may suppress the accent, if we remember that the origin is now at the centre C, and thus write the equation

$$y^2 = (e^2-1)(x^2 - a^2) \text{ .... } (2).$$

In (2) suppose $x = 0$, then $y^2 = -(e^2-1)\,a^2$; this gives an impossible value to y, and thus the curve does not cut the axis of y. We shall however denote $(e^2-1)\,a^2$ by b^2, and measure off the ordinates CB and CB' each equal to b, as we shall find these ordinates useful hereafter.

Thus (1) may be written

$$y^2 = \frac{b^2}{a^2}(2ax + x^2) \text{ .... } (3),$$

and (2) may be written

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2). \qquad (4),$$

or, more symmetrically,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \text{or, } a^2y^2 - b^2x^2 = -a^2b^2 \ldots\ldots\ldots (5).$$

212. Since $AH = eOA$ and $OA = \frac{p}{1+e}$, we have

$$AH = \frac{ep}{1+e} = \frac{(e-1)\,ep}{e^2-1} = (e-1)\,a,$$

$$OA = \frac{p}{1+e} = \frac{e-1}{e}a, \quad (1 - \tfrac{1}{e})\,a$$

$$CH = CA + AH = a + (e-1)\,a = ea,$$

$$CO = CA - OA = a - \frac{\quad}{e}a = \frac{\ }{e},$$

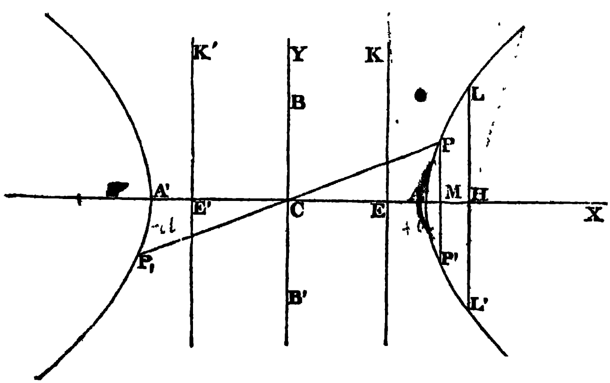
and

$$OH = p = \frac{a\,(e^2-1)}{e}.$$

213. We may now ascertain the form of the hyperbola. Take the equation referred to the centre as origin,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2). \qquad (1).$$

For every value of x less than a, y is impossible. When $x = a$, $y = 0$. For every value of x greater than a there



are two values of y equal in magnitude but of opposite sign. Hence if P be a point in the curve on one side of the axis of x, there is a point P' on the other side of the axis, such that $P'M = PM$. Hence the curve is symmetrical with respect to the axis of x, and it extends indefinitely to the right of A.

If we ascribe to x any negative value we obtain for y the same pair of values as when we ascribed to x the corresponding positive value. Hence the portion of the curve to the left of the axis of y is similar to the portion to the right of it.

As the equation (1) may be put in the form

$$x^2 = \frac{a^2}{b^2}(y^2 + b^2). \qquad (2),$$

we see that the axis of y also divides the curve symmetrically, and that the curve extends above and below CA. Thus the curve consists of two similar branches each extending indefinitely.

The line EK is the directrix, H is the corresponding focus. Since the curve is symmetrical with respect to the line BCB', it follows that if we take $CS = CH$ and $CE' = CE$, and draw $E'K'$ perpendicular to CE', the point S and the line $E'K'$ will form respectively a second focus and directrix, by means of which the curve might have been generated.

214. The point C is called the *centre* of the hyperbola, because every chord of the hyperbola which passes through C is bisected in C. This is proved in the same manner as the corresponding proposition in the ellipse. (See Art. 163.)

215. We have drawn the curve concave towards the axis of x; the following proposition will justify the figure.

The ordinate of any point of the curve which lies between a vertex and a fixed point of the curve on the same branch as the vertex is greater than the corresponding ordinate of the straight line joining that vertex and the fixed point.

Let A be the vertex and take it for the origin; let P be the fixed point; x', y' its co-ordinates. Then the equation to the hyperbola is (Art. 211)

$$y^2 = \frac{b^2}{a^2}(2ax + x^2).$$

The equation to AP is

$$y = \frac{y'}{x'}x,$$

or
$$y = \frac{b}{a}\sqrt{\left(\frac{2a}{x'} + 1\right)}x,$$

since (x', y') is on the hyperbola.

Let x denote any abscissa less than x', then since the ordinate of the curve is $\frac{b}{a}\sqrt{(2ax + x^2)}$ or $\frac{b}{a}\sqrt{\left(\frac{2a}{x} + 1\right)}x$, and that of the straight line is $\frac{b}{a}\sqrt{\left(\frac{2a}{x'} + 1\right)}x$, it is obvious that the ordinate of the curve is greater than that of the line.

216. AA' and BB' are called *axes* of the hyperbola. The axis AA' which if produced passes through the foci, is called the *transverse* axis, and BB' the *conjugate* axis. We do not as in the case of the ellipse, use the terms *major* and *minor* axis, because since $b = a\sqrt{(e^2 - 1)}$ (Art. 211), and e is greater than unity, b may be greater or less than a.

The ratio which the distance of any point on the hyperbola from the focus bears to the distance of the same point from the corresponding directrix is called the *excentricity* of the hyperbola. We have denoted it by the symbol e.

To find the *latus rectum* (see Art. 128) we put $x = CH$, that is, $= ae$, in equation (1) of Art. 213; thus

$$y^2 = \frac{b^2a^2(e^2 - 1)}{a^2} = \frac{b^4}{a^2};$$

$$\therefore\ LH = \frac{b^2}{a}, \text{ and the latus rectum} = \frac{2b^2}{a}$$

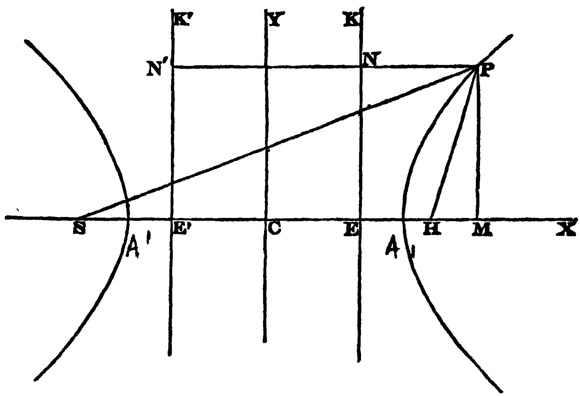
Since $b^2 = a^2(e^2-1)$; $\therefore\ b^2 + a^2 = a^2e^2$; that is,

$$CB^2 + CA^2 = CH^2;$$

$$\therefore\ AB = CH.$$

217. The equation to the hyperbola may be derived from the equation to the ellipse by writing $-b^2$ for b^2. We shall find that the hyperbola has many properties similar to those which have been proved for the ellipse; and as the demonstrations are similar to those which have been given, we shall in some cases not repeat them for the hyperbola, but refer to the corresponding articles in the chapters on the ellipse.

218. *To express the focal distances of any point of the hyperbola in terms of the abscissa of the point.*



Let S be one focus, $E'K'$ the corresponding directrix; H the other focus, EK the corresponding directrix. Let P be a point on the hyperbola; x, y its co-ordinates, the centre being the origin. Join SP, HP, and draw PNN' parallel to the transverse axis and PM perpendicular to it. Then

$$SP = ePN' = e\,(CM + CE') = e\left(x + \frac{a}{e}\right) = ex + a,$$

$$HP = ePN = e\,(CM - CE) = e\left(x - \frac{a}{e}\right) = ex - a.$$

Hence $SP - HP = 2a$; that is, the *difference* of the focal distances of any point on the hyperbola is equal to the transverse axis.

219. The equation $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$ may be written

$$y^2 = \frac{b^2}{a^2}(x-a)(x+a).$$

Hence (see Fig. to Art. 213),

$$\frac{PM^2}{AM \cdot A'M} = \frac{BC^2}{AC^2}.$$

Tangent and Normal to an Hyperbola.

220. *To find the equation to the tangent at any point of an hyperbola.*

By a process similar to that in Art. 170, it will be found that the equation to the tangent at the point (x', y') is

$$y - y' = \frac{b^2 x'}{a^2 y'}(x - x'),$$

or

$$a^2 yy' - b^2 xx' = -a^2 b^2.$$

These equations may be derived from the corresponding equations with respect to the ellipse by writing $-b^2$ for b^2.

221. The equation to the tangent to the hyperbola may also be written in the form (see Art. 171)

$$y = mx + \sqrt{(m^2 a^2 - b^2)}.$$

Conversely every line whose equation is of this form is a tangent to the hyperbola.

222. It may be shewn as in the case of the circle that a tangent to an hyperbola meets it in only *one* point. Also if a

line meet an hyperbola in only *one* point, it is in general the tangent to the hyperbola at that point. For suppose

$$a^2y^2 - b^2x^2 = -a^2b^2$$

to be the equation to an hyperbola, and

$$y = mx + c$$

the equation to a straight line. Then to determine the abscissæ of the points of intersection, we have the equation

$$a^2(mx + c)^2 - b^2x^2 = -a^2b^2,$$

or $$(a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(c^2 + b^2) = 0.$$

This equation has always two roots, except

(1) when $$a^4m^2c^2 = (a^2m^2 - b^2)a^2(c^2 + b^2),$$

or $$c^2 = m^2a^2 - b^2,$$

and consequently the line is a tangent;

(2) when $a^2m^2 - b^2 = 0$; the equation then reduces to one of the first degree, and therefore has but one root. Thus a line which meets the hyperbola in *one* point only is the tangent at that point unless the inclination of the line to the transverse axis be $\pm \tan^{-1} b$

223. The tangents at the vertices A and A' are parallel to the axis of y. (See Art. 172.)

224. *To find the equation to the normal at any point of an hyperbola.* (See Art. 173.)

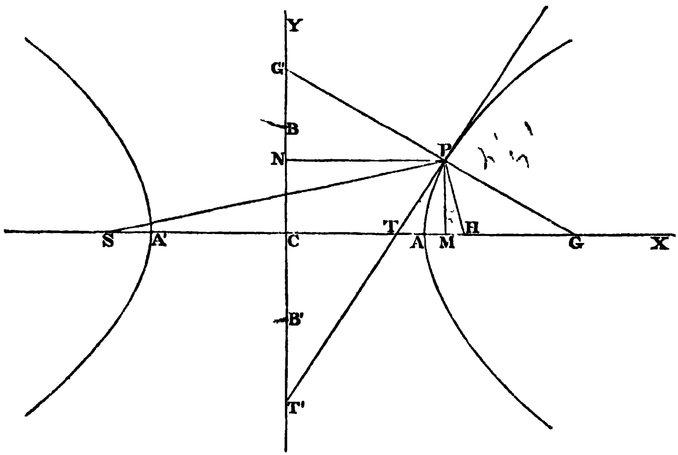
It will be found that the equation to the normal at (x', y') is

$$y - y' = -\frac{a^2y'}{b^2x'}(x - x').$$

This may also be written in the form

$$y = mx - \frac{(a^2+b^2)m}{\sqrt{(a^2-b^2m^2)}}. \quad \text{(See Art. 174.)}$$

225. We shall now deduce some properties of the hyperbola from the preceding articles.



Let x', y' be the co-ordinates of P; let PT be the tangent at P, PG the normal at P; PM, PN perpendiculars on the axes.

The equation to the tangent at P is

$$a^2yy' - b^2xx' = -a^2b^2.$$

Let $y=0$, then $x = \frac{a^2}{x'}$, hence

$$CT = \frac{CA^2}{CM},$$

$$\therefore CM \, . \, CT = CA^2.$$

Similarly $$CN \, . \, CT' = CB^2.$$

226. As in Art. 176, we may shew that

$$CG = e^2 CM,$$

$$CG' = \frac{a^2e^2}{b^2} PM.$$

227. As in Art. 177, we may shew that

$$PG^2 = \frac{b^2rr'}{a^2}, \quad PG'^2 = \frac{a^2rr'}{b^2};$$

where $$SP = r', \; HP = r.$$

228. *The tangent at any point bisects the angle between the focal distances of that point.*

For in the manner given in Art. 178, we may shew that

$$SPG' = HPG;$$

and therefore since PT is perpendicular to GG',

$$TPS = TPH.$$

Or we may prove the result thus,

$$CG = e^2x' \text{ (Art. 226)};$$

$$\therefore \; SG = e^2x' + ae,$$

$$HG = e^2x' - ae.$$

Also $SP = ex' + a$, $HP = ex' - a$; hence

$$\frac{SG}{HG} = \frac{SP}{HP};$$

therefore by Euclid, VI. 3, PG bisects the angle between HP and SP produced, that is,

$$SPG' = HPG.$$

229. *To find the locus of the intersection of the tangent at any point with the perpendicular on it from the focus.*

It may be proved as in Art. 180, that the required locus is the circle described on the transverse axis as diameter.

230. Let p denote the perpendicular from H on the tangent at P, and p' the perpendicular from S; then, as in Art. 181, it may be shewn that

$$p^2 = \frac{b^2 r}{r'}, \quad p'^2 = \frac{b^2 r'}{r}; \quad \therefore pp' = b^2.$$

Since $\qquad r' = 2a + r$, we have

$$p^2 = \frac{}{2a + r}.$$

231. *From any external point two tangents can be drawn to an hyperbola.*

Let h, k be the co-ordinates of the external point, then as in Art. 182, we shall obtain the following equation for determining the abscissæ of the points of contact of the tangents and hyperbola,

$$x'^2 (a^2k^2 - b^2h^2) + 2a^2b^2hx' - a^4 (b^2 + k^2) = 0.$$

The roots of this quadratic will be possible if

$$a^4b^4h^2 + a^4 (b^2 + k^2) (a^2k^2 - b^2h^2) \text{ is positive;}$$

that is, if

$$k^2a^2 - b^2h^2 + a^2b^2$$

is positive.

But if (h, k) be an *external* point the last expression *is* positive, and therefore two tangents can be drawn to the hyperbola from an external point.

The product of the two values of x' given by the above quadratic is

$$-\frac{a^4 (b^2 + k^2)}{a^2k^2 - b^2h^2};$$

this product is therefore positive or negative according as $a^2k^2 - b^2h^2$ is negative or positive; that is, the two tangents meet the *same* branch or *different* branches according as $a^2k^2 - b^2h^2$ is negative or positive.

The case in which $a^2k^2 - b^2h^2 = 0$ requires to be noticed. Here one root of the quadratic equation becomes infinite, and the other is $\frac{a^4(b^2+k^2)}{2a^2b^2h}$; see *Algebra*, Chapter XXII.

In this case the point (h, k) falls on a certain line called an *asymptote*, which we shall consider hereafter; see Art. 255. The asymptote itself may then count as one of the two tangents from the point (h, k). If $h = 0$ and $k = 0$ the point (h, k) is the origin; in this case the two asymptotes may count as the two tangents from the point (h, k).

232. *Tangents are drawn to an hyperbola from a given external point; to find the equation to the chord of contact.*

Let h, k be the co-ordinates of the external point; then the equation to the chord of contact is

$$a^2ky - b^2hx = -a^2b^2. \quad \text{(See Art. 183.)} \quad -\frac{xh}{a^2} + \frac{yk}{b^2}$$

233. *Through any fixed point chords are drawn to an hyperbola, and tangents to the hyperbola are drawn at the extremities of each chord; the locus of the intersection of the tangents is a straight line.*

Let h, k be the co-ordinates of the point through which the chords are drawn, then the equation to the locus of the intersection of the tangents is

$$a^2ky - b^2hx = -a^2b^2. \quad \text{(See Art. 184.)}$$

234. *If from any point in a straight line a pair of tangents be drawn to an hyperbola, the chords of contact will all pass through a fixed point.* (See Art. 185.)

235. The student should observe the different interpretations that can be assigned to the equation

$$a^2ky - b^2hx = -a^2b^2.$$

The statements in Art. 103 with respect to the circle may all be applied to the hyperbola.

X /

EXAMPLES.

1. What is the equation to an hyperbola of given transverse axis whose vertex bisects the distance between the centre and focus?

2. If the ordinate MP of an hyperbola be produced to Q so that $MQ = SP$, find the locus of Q.

3. Any chord AP through the vertex of an hyperbola is divided in Q so that $AQ : QP :: AC^2 : BC^2$, and QM is drawn to the foot of the ordinate MP; from Q a line is drawn at right angles to QM meeting the transverse axis in O; shew that $AO : A'O :: AC^2 : BC^2$.

4. PQ is a chord of an ellipse at right angles to the major axis AA'; PA, QA' are produced to meet in R; shew that the locus of R is an hyperbola having the same axes as the ellipse.

5. If a circle be described passing through any point P of a given hyperbola and the extremities of the transverse axis, and the ordinate MP be produced to meet the circle in Q, shew that the locus of Q is an hyperbola whose conjugate axis is a third proportional to the conjugate and transverse axes of the original hyperbola.

6. Find the locus of a point such that if from it a pair of tangents be drawn to an ellipse the product of the perpendiculars dropped from the foci upon the chord of contact will be constant.

CHAPTER XII.

THE HYPERBOLA CONTINUED.

Diameters.

236. *To find the length of a line drawn from any point in a given direction to meet an hyperbola.*

Let x', y' be the co-ordinates of the point from which the line is drawn; x, y the co-ordinates of the point to which the line is drawn; θ the inclination of the line to the axis of x; r the length of the line; then (Art. 27)

$$x = x' + r\cos\theta, \qquad y = y' + r\sin\theta \ldots\ldots\ldots\ldots (1).$$

If (x, y) be on the hyperbola these values may be substituted in the equation $a^2y^2 - b^2x^2 = -a^2b^2$; thus

$$a^2(y' + r\sin\theta)^2 - b^2(x' + r\cos\theta)^2 = -a^2b^2;$$

$$\therefore\ r^2(a^2\sin^2\theta - b^2\cos^2\theta) + 2r(a^2y'\sin\theta - b^2x'\cos\theta)$$
$$+ a^2y'^2 - b^2x'^2 + a^2b^2 = 0 \ldots\ldots\ldots\ldots (2).$$

From this quadratic two values of r can be found which are the lengths of the two lines that can be drawn from (x', y') in the given direction to the hyperbola.

locus of the middle points

237. *To find the diameter of a given system of parallel chords in an hyperbola.* (See definition in Art. 148.)

Let θ be the inclination of the chords to the transverse axis of the hyperbola; let x', y' be the co-ordinates of the middle point of any one of the chords; the equation which determines the lengths of the lines drawn from (x', y') to the curve is (Art. 236)

$$r^2(a^2\sin^2\theta - b^2\cos^2\theta) + 2r(a^2y'\sin\theta - b^2x'\cos\theta)$$
$$+ a^2y'^2 - b^2x'^2 + a^2b^2 = 0 \ldots\ldots\ldots\ldots (1).$$

Since (x', y') is the middle point of the chord, the values of r furnished by this equation must be *equal in magnitude and opposite in sign* hence the coefficient of r must vanish; thus

$$a^2y' \sin\theta - b^2x' \cos\theta = 0, \quad \text{or } y' = \frac{b^2}{a^2}\cot\theta \,.\, x'. \qquad (2).$$

Considering x' and y' as variable this is the equation to a straight line passing through the origin, that is, through the centre of the hyperbola.

Hence every diameter passes through the centre.

Also every straight line passing through the centre is a diameter, that is, bisects some system of parallel chords. For by giving to θ a suitable value the equation (2) may be made to represent *any* line passing through the centre. If θ' be the inclination to the axis of x of the diameter which bisects all the chords inclined at an angle θ, we have from (2)

$$\tan\theta' = \frac{b^2}{a^2}\cot\theta;$$

$$\therefore \tan\theta\tan\theta' = \frac{b^2}{a^2}. \qquad (3).$$

238. *If one diameter bisect all chords parallel to a second diameter, the second diameter will bisect all chords parallel to the first.*

Let θ_1 and θ_2 be the respective inclinations of the two diameters to the transverse axis of the hyperbola. Since the first bisects all the chords parallel to the second, we have

$$\tan\theta_2\tan\theta_1 = \frac{b^2}{a^2}.$$

And this is also the only condition that must hold in order that the second may bisect the chords parallel to the first.

The definition in Art. 191 holds for the hyperbola.

239. Every straight line passing through the centre of an ellipse meets that ellipse; this is evident from the figure, or it may be proved analytically. But in the case of an hyperbola this proposition is not true, as we proceed to shew.

240. To find the points of intersection of an hyperbola with a straight line passing through its centre.

Let the equation to the straight line be

$$y = mx.$$

Substitute this value of y in the equation to the hyperbola

$$a^2y^2 - b^2x^2 = -a^2b^2;$$

then we have for determining the abscissæ of the points of intersection the equation

$$(a^2m^2 - b^2)\,x^2 = -a^2b^2;$$

$$\therefore x^2 = \frac{a^2b^2}{b^2 - a^2m^2}.$$

Hence the values of x are impossible if a^2m^2 is greater than b^2.

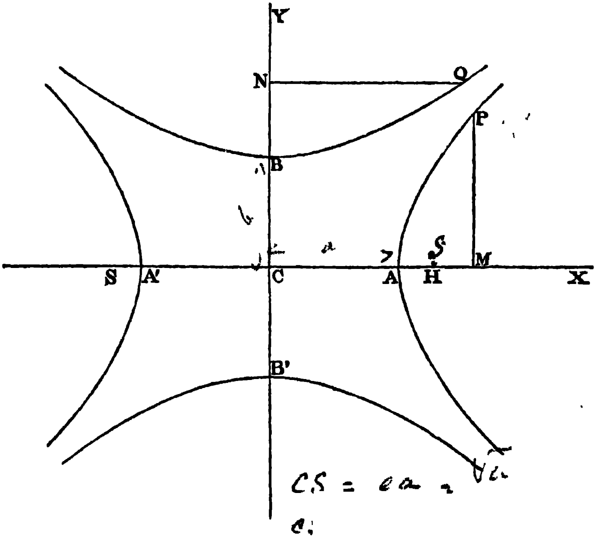
Thus a line drawn through the centre of an hyperbola will not meet the curve if it makes with the transverse axis on either side of it an angle greater than $\tan^{-1}\frac{b}{a}$.

241. It is convenient for the sake of enunciating many properties of the hyperbola to introduce the following important definition.

DEF. The conjugate hyperbola is an hyperbola having for its transverse and conjugate axes the conjugate and transverse axes of the original hyperbola respectively.

242. *To find the equation to the hyperbola conjugate to a given hyperbola.*

Let AA', BB' be the transverse and conjugate axes respectively of the given hyperbola; then BB' is the transverse axis of the conjugate hyperbola, and AA' is its conjugate axis. Let P be a point in the given hyperbola, Q a point in the conjugate hyperbola. Draw PM, QN perpendicular to



CX, CY respectively. The equation to the given hyperbola is

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2);$$

$$\therefore\ PM^2 = \frac{CB^2}{CA^2}(CM^2 - CA^2).$$

Hence $$QN^2 = \frac{CA^2}{CB^2}(CN^2 - CB^2),$$

since Q is a point on an hyperbola having CB, CA for its semi-transverse and semi-conjugate axes respectively. Thus if x, y denote the co-ordinates of Q,

$$x^2 = \frac{a^2}{b^2}(y^2 - b^2).$$

This, therefore, is the equation to the conjugate hyperbola; we observe that it may be deduced from the equation to the given hyperbola by writing $-a^2$ for a^2 and $-b^2$ for b^2.

The foci of the conjugate hyperbola will be on the line BCB' at a distance from $C = AB$ (Art. 216); that is, at the same distance from C as S and H.

243. *Every straight line drawn through the centre of an hyperbola meets the hyperbola or the conjugate hyperbola, except the two lines inclined to the transverse axis of the hyperbola at an angle* $= \tan^{-1}\dfrac{b}{a}$.

Let the equation to the straight line be

$$y = mx \quad \text{..............................} \quad (1).$$

To find the abscissæ of the points of intersection of (1) with the given hyperbola, we have, as in Art. 240, the equation

$$x^2 = \frac{a^2b^2}{b^2 - a^2m^2} \quad \text{........................} \quad (2).$$

Similarly to find the points of intersection of (1) with the conjugate hyperbola, we have the equation

$$x^2 = \frac{a^2b^2}{a^2m^2 - b^2} \quad \text{........................} \quad (3).$$

If m^2 be less than $\dfrac{b^2}{a^2}$, (2) gives possible values, and (3) impossible values for x; if m^2 be greater than $\dfrac{b^2}{a^2}$, (2) gives impossible values, and (3) possible values for x; if $m^2 = \dfrac{b^2}{a^2}$, (2) and (3) make x infinite. Thus the two lines that can be drawn at an inclination $\tan^{-1}\dfrac{b}{a}$ to the transverse axis of the given hyperbola meet neither curve; and every other line meets one of the curves.

244. *Of two conjugate diameters one meets the original hyperbola, and the other the conjugate hyperbola.*

Let the equations to the two diameters be

$$y = mx, \qquad y' = m'x;$$

then, by Art. 238, $mm' = \frac{b^2}{a^2}$; $m^2m'^2 = \frac{b^4}{a^4}$.

Hence if m^2 is less than $\frac{b^2}{a^2}$, m'^2 is greater than $\frac{b^2}{a^2}$; thus the first diameter meets the original hyperbola, and the second the conjugate hyperbola. If m^2 is greater than $\frac{b^2}{a^2}$, m'^2 is less than $\frac{b^2}{a^2}$; thus the first diameter meets the conjugate hyperbola, and the second the original hyperbola.

245. We proceed now to some properties connected with conjugate diameters. When we speak of the *extremities* of a diameter we mean the points where that diameter intersects the original hyperbola or the conjugate hyperbola.

We may remark that the original hyperbola bears the same relation to the conjugate hyperbola as the conjugate hyperbola bears to the original hyperbola. Thus the definition may be given as follows: two hyperbolas are called conjugate when each has for its transverse axis the conjugate axis of the other.

Also if a line bisect all parallel chords terminated by one of the hyperbolas it bisects all the chords of the same system which are terminated by the other hyperbola. For the equation (Art. 237) $\tan\theta \tan\theta' = \frac{b^2}{a^2}$ remains unchanged when we write $-a^2$ for a^2 and $-b^2$ for b^2, that is, when we pass from the original hyperbola to the conjugate (Art. 242).

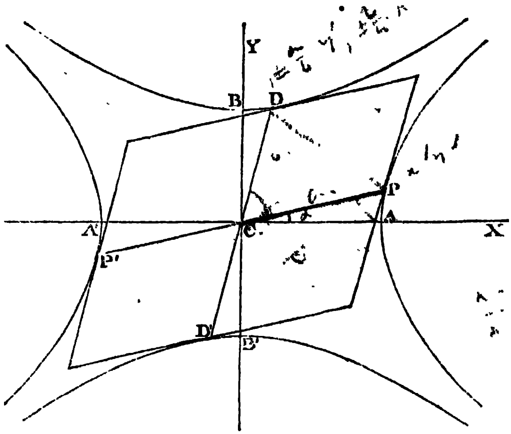
Both curves are comprised in the equation

$$(a^2y^2 - b^2x^2)^2 = a^4b^4.$$

246. *The tangent at either extremity of any diameter is parallel to the chords which that diameter bisects.* See Art. 190.

247. *Given the co-ordinates of one extremity of a diameter, to find those of either extremity of the conjugate diameter.*

Let ACA', BCB' be the axes of an hyperbola; PCP',



DCD' a pair of conjugate diameters. Let x', y' be the given co-ordinates of P; then the equation to CP is

$$y = \frac{y'}{x'} x. \qquad (1).$$

Since the conjugate diameter DD' is parallel to the tangent at P, the equation to DD' is

$$y = \frac{b^2 x'}{a^2 y'} x \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (2).$$

We must combine (2) with the equation to the conjugate hyperbola to find the coördinates of D and D'. Substitute from (2) in

$$a^2y^2 - b^2x^2 = a^2b^2; \text{ then}$$

$$a^2 \frac{b^4}{a^4} \frac{x'^2}{y'^2} x^2 - b^2x^2 = a^2b^2;$$

$$\therefore \; (b^2x'^2 - a^2y'^2)\, x^2 = a^4y'^2;$$

$$\therefore \; x^2 = \frac{a^4y'^2}{a^2b^2} = \frac{a^2y'^2}{b^2};$$

$$\therefore \; x = \pm \frac{ay'}{b};$$

$$\therefore \text{ from (2), } y = \pm \frac{bx'}{a}.$$

In the figure the abscissa of D is positive, and that of D' negative; hence the upper sign applies to D, and the lower to D'.

248. *The difference of the squares of two conjugate semi-diameters is constant.*

Let x', y' be the co-ordinates of P; then, by the preceding article,

$$CP^2 - CD^2 = x'^2 + y'^2 - \frac{a^2y'^2}{b^2} - \frac{b^2x'^2}{a^2}$$

$$= \frac{b^2x'^2 - a^2y'^2}{b^2} + \frac{a^2y'^2 - b^2x'^2}{a^2}$$

$$= a^2 - b^2.$$

Hence the difference of the squares of two conjugate semi-diameters is equal to the difference of the squares of the semi-axes.

249. *The area of the parallelogram formed by tangents at the ends of conjugate diameters is constant.*

Let PCP', DCD' be the conjugate diameters (see Fig. to Art. 247). The area of the parallelogram formed by tangents at P, D, P', D', is $4CP.CD \sin PCD$, or $4p.CD$, where p denotes the perpendicular from C on the tangent at P. Let x', y' be the co-ordinates of P; then the equation to the tangent at P is

$$y = \frac{b^2x'}{a^2y'}x - \frac{b^2}{y'}$$

Hence (Art. 47)

$$p = \frac{\frac{b^2}{y'}}{\sqrt{\left(1 + \frac{b^4x'^2}{a^4y'^2}\right)}} = \frac{a^2b^2}{\sqrt{(a^4y'^2 + b^4x'^2)}}.$$

And $$CD = \sqrt{\left(\frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2}\right)} = \frac{\sqrt{(a^4y'^2 + b^4x'^2)}}{ab};$$

$$\therefore\ 4p \cdot CD = 4ab.$$

Hence the area of any parallelogram formed by tangents at the ends of conjugate diameters is equal to the area of the rectangle formed by tangents at the ends of the axes.

250. Let a', b' denote the lengths of two conjugate semi-diameters; α the angle between them; by the preceding article,

$$a'b' \sin \alpha = ab.$$

By making P move along the hyperbola from A we can make CP or a' as great as we please. Also since $a'^2 - b'^2$ is constant, b' increases with a'. Thus $\sin \alpha$ can be made as small as we please, that is, CP and CD can be brought as near to coincidence as we please. The limiting position towards which they tend is easily found; for from Art. 237,

$$mm' = \frac{-}{a^2};$$

thus the limit to which m and m' approach as CP and CD approach to coincidence is $\pm \frac{b}{a}$.

251. From Art. 249 we have

$$p^2 = \frac{a^2b^2}{CD^2} = \frac{a^2b^2}{CP^2 - a^2 + b^2}. \quad \text{(Art. 248.)}$$

This gives a relation between p the perpendicular from the centre on the tangent at any point P, and the distance CP of that point from the centre.

Also if ϕ denote the angle which the perpendicular makes with the transverse axis, we may shew as in Art. 196 that

$$p^2 = a^2 (1 - e^2 \sin^2 \phi).$$

252. *To find the equation to the hyperbola referred to a pair of conjugate diameters as axes.*

Let CP, CD be two conjugate semidiameters (see Fig. to Art. 247), take CP as the new axis of x, CD as that of y; let $PCA = \alpha$, $DCA = \beta$. Let x, y be the co-ordinates of any point of the hyperbola referred to the original axes; x', y' the co-ordinates of the same point referred to the new axes; then (Art. 84)

$$x = x' \cos\alpha + y' \cos\beta,$$

$$y = x' \sin\alpha + y' \sin\beta.$$

Substitute these values in the equation

$$a^2y^2 - b^2x^2 = -a^2b^2;$$

then $\quad a^2(x'\sin\alpha + y'\sin\beta)^2 - b^2(x'\cos\alpha + y'\cos\beta)^2 = -a^2b^2,$

or $\quad x'^2(a^2\sin^2\alpha - b^2\cos^2\alpha) + y'^2(a^2\sin^2\beta - b^2\cos^2\beta)$

$$+ 2x'y'(a^2\sin\alpha\sin\beta - b^2\cos\alpha\cos\beta) = -a^2b^2.$$

But since CP and CD are conjugate semidiameters,

$$\tan\alpha\tan\beta = \frac{b^2}{a^2};$$

hence the coefficient of $x'y'$ vanishes, and the equation becomes

$$x'^2(a^2\sin^2\alpha - b^2\cos^2\alpha) + y'^2(a^2\sin^2\beta - b^2\cos^2\beta) = -a^2b^2.$$

In this equation suppose $y' = 0$, then

$$x'^2 = \frac{-a^2b^2}{a^2\sin^2\alpha - b^2\cos^2\alpha} = \frac{a^2b^2}{b^2\cos^2\alpha - a^2\sin^2\alpha}.$$

This is the value of CP^2 which we shall denote by a'^2. If we put $x' = 0$ in the above equation, we obtain

$$y'^2 = \frac{-a^2b^2}{a^2\sin^2\beta - b^2\cos^2\beta}.$$

Now since we have supposed that the new axis of x meets the curve, we know that the new axis of y will *not* meet the curve (Art. 244), so that

$$\frac{-a^2b^2}{a^2\sin^2\beta - b^2\cos^2\beta}$$

is not a *positive* quantity; we shall denote it by $-b'^2$. Hence the equation to the hyperbola referred to conjugate diameters is

$$\frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} = 1,$$

or, suppressing the accents on the variables,

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1.$$

Also the equation to the conjugate hyperbola referred to the same axes is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = -1.$$

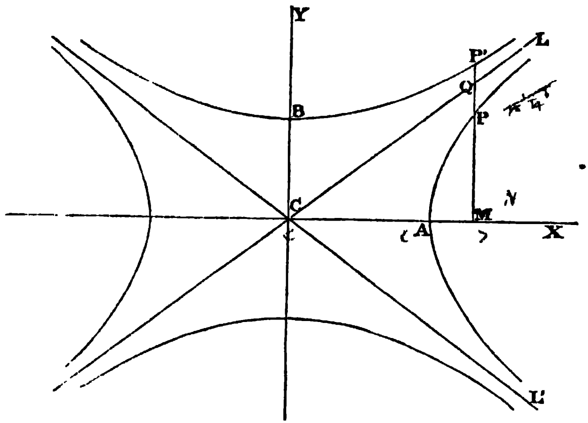
The equation to the tangent to the hyperbola will be of *the same form* whether the axes be rectangular or the oblique system formed by a pair of conjugate diameters. (See Art. 200.)

253. *Tangents at the extremities of any chord of an hyperbola meet in the diameter which bisects that chord.* (See Art. 201.)

254. *If a chord and diameter of an hyperbola are parallel, the supplemental chord is parallel to the conjugate diameter.* (See Arts. 202, 203.)

Asymptotes.

255. The properties of the hyperbola hitherto given have been similar to those of the ellipse; we have now to consider some properties peculiar to the hyperbola.



Let the equation to the hyperbola be

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2),$$

and let CL be the line which has for its equation

$$y = \frac{bx}{a}.$$

Let MPQ be an ordinate meeting the hyperbola in P and the straight line CL in Q; then if CM be denoted by x,

$$PM = \frac{b}{a}\sqrt{(x^2 - a^2)}, \quad QM = \frac{bx}{a};$$

$$\therefore PQ = \frac{b}{a}\{x - \sqrt{(x^2 - a^2)}\} = \left(\frac{b}{a} \cdot \frac{a^2}{x + \sqrt{(x^2 - a^2)}}\right) = \frac{ab}{x + \sqrt{(x^2 - a^2)}}.$$

If then the line MPQ be supposed to move parallel to itself from A, the distance PQ continually diminishes, and by taking

CM large enough we may make PQ as small as we please. The line CL is called an *asymptote* of the curve.

Similarly the line CL', which has for its equation

$$y = -\frac{bx}{a},$$

is an asymptote.

Thus the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

includes both asymptotes. We may take the following definition.

Def. An asymptote is a straight line the distance of which from a point of a curve diminishes without limit as the point in the curve moves to an infinite distance from the origin.

The distance of P from CL is $PQ \sin PQC$; and as we have seen that PQ diminishes without limit as P moves away from the origin, CL is an *asymptote* according to the definition here given.

256. In the same manner we may shew that CL is an asymptote to the conjugate hyperbola. For let MP be produced to meet the conjugate hyperbola in P', then (Art. 242)

$$P'M = \frac{b}{a}\sqrt{(x^2 + a^2)};$$

$$\therefore\ P'Q = \frac{b}{a}\{\sqrt{(x^2 + a^2)} - x\} = \frac{ba}{\sqrt{(x^2 + a^2)} + x}.$$

Hence as CM is increased indefinitely $P'Q$ is diminished indefinitely; therefore CL is an asymptote to the conjugate hyperbola.

257. The equation to the tangent to the hyperbola at the point (x', y') is

$$a^2yy' - b^2xx' = -a^2b^2,$$

$$y = \frac{b^2x'x}{a^2y'} - \frac{b^2}{y'}$$

$$= \frac{b}{a}\cdot\frac{x'x}{\sqrt{(x'^2 - a^2)}} - \frac{b^2}{y'}$$

$$= \frac{bx}{a\sqrt{\left(1 - \frac{a^2}{x'^2}\right)}} - \frac{b^2}{y'}.$$

If x' and y' are increased indefinitely the limiting form to which the above equation approaches is

$$y = \frac{bx}{a}.$$

Thus the tangent to the hyperbola approaches continually to coincidence with an asymptote when the point of contact moves away indefinitely from the origin.

258. It appears from Art. 243 that every straight line drawn through the centre of an hyperbola must meet the hyperbola or its conjugate, unless its direction coincides with that of one of the asymptotes. And from Art. 250 it appears that as conjugate diameters increase indefinitely they approach to coincidence with one of the asymptotes.

259. *The line joining the ends of conjugate diameters is parallel to one asymptote and bisected by the other.*

Let x', y' be the co-ordinates of any point P on the hyperbola (see Fig. to Art. 247); then the co-ordinates of D the extremity of the conjugate diameter are (Art. 247)

$$\frac{ay'}{b} \text{ and } \frac{bx'}{a}$$

Hence the equation to DP is

$$y - y' = \frac{y' - \frac{bx'}{a}}{x' - \frac{ay'}{b}}(x - x'),$$

that is, $$y - y' = -\frac{b}{a}(x - x');$$

and therefore DP is parallel to the asymptote

$$y = -\frac{bx}{a}.$$

Also the co-ordinates of the middle point of DP are (Art. 10)

$$\frac{1}{2}\left(x' + \frac{ay'}{b}\right) \text{ and } \frac{1}{2}\left(y' + \frac{bx'}{a}\right),$$

that is, $$\frac{ay' + bx'}{2b} \text{ and } \frac{ay' + bx'}{2a}.$$

These co-ordinates satisfy the equation

$$y = \frac{bx}{a};$$

therefore the asymptote $y = \frac{bx}{a}$ bisects PD.

Since the diagonals of a parallelogram bisect each other, and PD is one diagonal of the parallelogram of which CP and CD are adjacent sides, the other diagonal coincides with the asymptote, that is, the tangents at P and D meet on the asymptote.

260. The equation to the hyperbola referred to conjugate diameters as axes is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1 \quad \text{......................... (1).}$$

Hence the equations to the asymptotes referred to these axes are

$$y = \frac{b'x}{a'}, \quad y = -\frac{b'x}{a'} \quad \text{..................... (2).}$$

For we may shew as in Art. 243 that the lines denoted by (2) are the only lines through the centre which meet neither (1) nor its conjugate. Hence these lines are the asymptotes by Art. 258.

Or the same conclusion may be obtained thus; the original equation to the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and that to the two asymptotes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

If by substituting for x and y their values in terms of the new co-ordinates x' and y', and suppressing accents on the variables, the former equation is reduced to

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1,$$

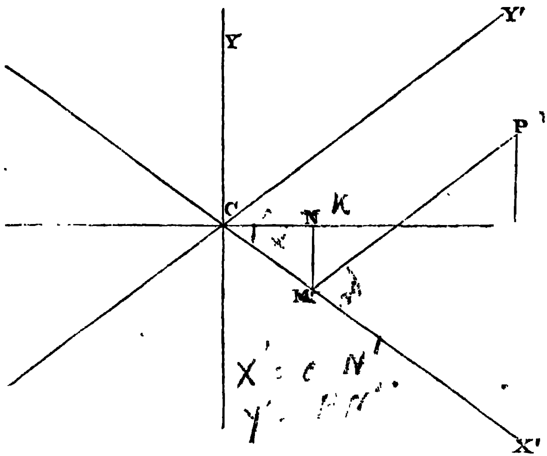
the latter must become, by the same substitution,

$$\frac{y^2}{b'^2} = 0.$$

261. *To find the equation to the hyperbola referred to the asymptotes as axes.*

Let CX, CY be the original axes; CX', CY' the new axes, so that CX' and CY' are inclined to CX on opposite sides of it at an angle α such that $\tan\alpha = \frac{b}{a}$. Let x, y be the co-ordinates of a point P referred to the old axes; x', y' the co-ordinates of the same point referred to the new axes. Draw PM' parallel to CY', and PM and $M'N$ each parallel to CY. Then

$$x = CM = CN + NM$$
$$= (x' + y') \cos\alpha.$$



So $$y = PM = (y' - x') \sin \alpha.$$

Also $\cos \alpha = \frac{a}{\sqrt{(a^2+b^2)}}$, $\sin \alpha = \frac{b}{\sqrt{(a^2+b^2)}}$; substitute these values in the equation

$$a^2y^2 - b^2x^2 = -a^2b^2;$$

then $$a^2b^2(y'-x')^2 - a^2b^2(y'+x')^2 = -a^2b^2(a^2+b^2),$$

or $$x'y' = \frac{a^2+b^2}{4},$$

or, suppressing the accents,

$$xy = \frac{a^2+b^2}{\ }$$

The equation to the conjugate hyperbola referred to the same axes is (Art. 242)

$$xy = -\frac{a^2+b^2}{4}.$$

262. *To find the equation to the tangent at any point of an hyperbola when the curve is referred to its asymptotes as axes.*

Let x', y' be the co-ordinates of the point;

x'', y'' the co-ordinates of an adjacent point on the curve. The equation to the secant through these points is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \quad\ldots\ldots\ldots\ldots\ldots (1).$$

Since (x', y') and (x'', y'') are points on the hyperbola

$$x'y' = \tfrac{1}{4}(a^2 + b^2),$$

$$x''y'' = \tfrac{1}{4}(a^2 + b^2);$$

$$\therefore\ x''y'' = x'y'.$$

Hence (1) may be written

$$y - y' = \frac{\dfrac{x'y'}{x''} - y'}{x'' - x'}(x - x'),$$

or $$y - y' = -\frac{y'}{x''}(x - x').$$

Now in the limit $x'' = x'$; hence the equation to the tangent at the point (x', y') is

$$y - y' = -\frac{y'}{x'}(x - x') \qquad (2).$$

This equation may be simplified; multiply by x', thus

$$yx' + xy' = 2x'y' = \frac{a^2 + b^2}{2}.$$

263. To find where the tangent at (x', y') meets the axis of x put $y = 0$ in the equation

$$yx' + xy' = \frac{a^2 + b^2}{2},$$

thus $$x = \frac{a^2 + b^2}{2y'} = \frac{2x'y'}{y'} = 2x'.$$

Similarly to find where the tangent cuts the axis of y put $x=0$ in the equation; thus

$$y=\frac{a^2+b^2}{2x'}=\frac{2x'y'}{x'}=2y'.$$

Thus the product of the intercepts $=4x'y'=a^2+b^2$. The area of the triangle contained between the tangent at any point and the asymptotes is equal to the product of the intercepts into half the sine of the included angle

$$=\tfrac{1}{2}(a^2+b^2)\sin 2\alpha=(a^2+b^2)\sin\alpha\cos\alpha=ab,$$

and is therefore constant.

Since the tangent at (x', y') cuts off intercepts $2x'$, $2y'$, from the axes of x and y respectively, the portion of the tangent at any point intercepted between the asymptotes is bisected at the point of contact.

Polar Equation.

264. To find the polar equation to the hyperbola, the focus being the pole.

Let $HP=r$, $AHP=\theta$; (see Fig. to Art. 209);

then $\qquad HP=ePN$, by definition;

that is, $\qquad HP=e(OH+HM)$;

or $\qquad r=a(e^2-1)+er\cos(\pi-\theta)$, (Art. 212);

$$\therefore\ r(1+e\cos\theta)=a(e^2-1),$$

and $$r=\frac{a(e^2-1)}{1+e\cos\theta} \quad\ldots\ldots\ldots\ldots (1).$$

If we denote the angle XHP by θ, then we have as before

$$HP=e(OH+HM);$$

thus $$r = a(e^2 - 1) + er\cos\theta,$$

and $$r = \frac{a(e^2-1)}{1 - e\cos\theta} \qquad (2).$$

We may also proceed thus; in the figure to Art. 218 suppose $SP = r$ and $PSH = \theta$:

then $$SP = ePN',$$

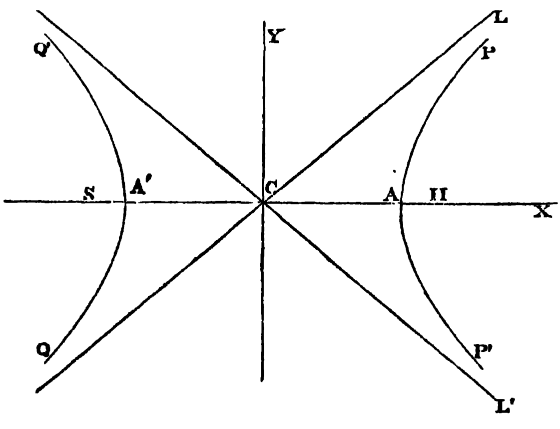
that is, $$SP = e(SM - SE');$$

or $$r = er\cos\theta - a(e^2-1);$$

$$\therefore\ r(e\cos\theta - 1) = a(e^2-1),$$

and $$r = \frac{a(e^2-1)}{e\cos\theta - 1} \qquad \text{.. } (3).$$

265. It will be a good exercise to trace the form of the hyperbola from any of these polar equations. Take for example the equation (1); suppose $\theta = 0$, then $r = a(e-1)$; we must therefore measure off the length $a(e-1)$ on the initial line from the pole H, and thus obtain the point A as one of the points of the curve.



As θ increases from 0 to $\frac{\pi}{2}$ we see from (1) that r increases; $\cos\theta$ is negative when θ is greater than $\frac{\pi}{2}$ and r continues to increase. Let α be such an angle that $1 + e\cos\alpha = 0$, that is, $\cos\alpha = -\frac{1}{e}$, then the nearer θ approaches to α the greater r becomes, and by taking θ near enough to α, we may make r as great as we please. Thus as θ increases from 0 to α that portion of the curve is traced out which begins at A and passes on through P to an indefinite distance from the origin.

When θ is greater than α, r is negative, and is at first indefinitely great and diminishes as θ increases from α to π. Since r is negative we measure it in the direction *opposite* to that we should use if it were positive. Thus as θ increases from α to π that portion of the curve is traced out which begins at an indefinite distance from C in the lower left-hand quadrant, and passes on through Q to A'. HA' is found by putting $\theta = \pi$ in (1); then r becomes $-a(e+1)$, therefore HA' is in length $= a(e+1)$.

As θ increases from π to $2\pi - \alpha$, r continues negative and numerically increases, and may be made as great as we please by taking θ sufficiently near to $2\pi - \alpha$. Thus the branch of the curve is traced out which begins at A' and passes on through Q' to an indefinite distance.

As θ increases from $2\pi - \alpha$ to 2π, r is again positive, and is at first indefinitely great and then diminishes. Thus the portion of the curve is traced out which begins at an indefinitely great distance from C in the lower right-hand quadrant and passes on through P' to A.

The asymptotes CL and CL' are inclined to the transverse axis at an angle of which the tangent is $\frac{b}{a}$; hence $\cos LCA$

$$= \frac{a}{\sqrt{(a^2+b^2)}} = \frac{1}{e}, \text{ and } \cos LCA' = -\frac{1}{e}; \text{ that is, } LCA' = \alpha.$$

Thus as θ approaches the value α the radius vector approaches to a position parallel to CL. Similarly as θ approaches the value $2\pi - \alpha$ the radius vector approaches to a position parallel to CL'.

266. As in Art. 205 it may be shewn that the polar equation to a chord subtending at the focus an angle 2β is

$$r = \frac{l}{e \cos\theta + \sec\beta \cos(\alpha - \theta)},$$

$\alpha - \beta$ and $\alpha + \beta$ being respectively the vectorial angles of the lines which join the focus to the ends of the chord, and l the semi-latus rectum.

Hence the polar equation to the tangent is

$$r = \frac{l}{e \cos\theta + \cos(\alpha - \theta)}.$$

267. The polar equation to the hyperbola, the centre being the pole, is (Art. 206)

$$r^2(a^2 \sin^2\theta - b^2 \cos^2\theta) = -a^2b^2.$$

Arts. 207, 208 are applicable to the Hyperbola.

Equilateral or Rectangular Hyperbola.

268. If in the equation to the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, we suppose $b = a$, we obtain $x^2 + y^2 = a^2$, which is the equation to a circle; so that the circle may be considered a particular case of the ellipse. If in the equation to the hyperbola $a^2y^2 - b^2x^2 = -a^2b^2$ we suppose $b = a$, we have $y^2 - x^2 = -a^2$. We thus obtain an hyperbola which is called the *equilateral* hyperbola from the equality of the axes. Since the angle between the asymptotes, which $= 2\tan^{-1}\frac{b}{a}$, becomes a right angle when $b = a$, the *equilateral* hyperbola is also called the *rectangular* hyperbola.

The peculiar properties of the rectangular hyperbola can be deduced from those of the ordinary hyperbola by making $b = a$.

Thus since $b^2 = a^2(e^2 - 1)$ we have $e^2 - 1 = 1$, $\therefore e = \sqrt{2}$.

The equation to the tangent is (Art. 220)

$$yy' - xx' = -a^2.$$

From Art. 227 $PG = PG' = \sqrt{(rr')}$.

Rectangular or equilateral

The equation to the conjugate hyperbola is, by Art. 242,

$$y^2 - x^2 = a^2.$$

Thus the conjugate hyperbola is the same curve as the original hyperbola, though differently situated.

By Art. 248, $CP = CD$, and therefore by Art. 259, CP and CD are equally inclined to the asymptotes.

EXAMPLES.

1. The radius of a circle which touches an hyperbola and its asymptotes is equal to that part of the latus rectum which is intercepted between the curve and asymptote.

2. A line drawn through one of the vertices of an hyperbola and terminated by two lines drawn through the other vertex parallel to the asymptotes will be bisected at the other point where it cuts the hyperbola.

3. If a straight line be drawn from the focus of an hyperbola the part intercepted between the curve and the asymptote $= \dfrac{a \sin \alpha}{\sin \alpha + \sin \theta}$, where θ and α are the angles made respectively by the straight line and asymptote with the axis.

4. PQ is one of a series of chords inclined at a constant angle to the diameter AB of a circle, find the locus of the point of intersection of AP and BQ.

5. P is a point in a branch of an hyperbola, P' is a point in a branch of its conjugate, CP, CP', being conjugate semi-diameters. If S, S' be the interior foci of the two branches, prove that the difference of SP and $S'P'$ is equal to the difference of AC and BC.

6. If x, y be co-ordinates of any point of an hyperbola, shew that we may assume $x = a \sec \theta$, $y = b \tan \theta$.

7. A line is drawn parallel to the axis of y meeting the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and its conjugate, in points P, Q; shew that the normals at P and Q intersect each other on the axis of x. Shew also that the tangents at P and Q intersect on the curve whose equation is $y^4(a^2y^2 - b^2x^2) = 4b^6x^2$.

8. Tangents to an hyperbola are drawn from any point in one of the branches of the conjugate; shew that the chord of contact will touch the other branch of the conjugate.

Find the equation to the radii from the centre to the points of contact of the two tangents, and if these radii are perpendicular to one another, shew that the co-ordinates of the point from which the tangents are drawn are

$$a\sqrt{\left(\frac{b^2 - 2a^2}{a^2 + b^2}\right)}, \quad b\sqrt{\left(\frac{2b^2 - a^2}{a^2 + b^2}\right)}.$$

9. Two tangents to a parabola include an angle α; shew that the locus of their point of intersection is an hyperbola with the same focus and directrix.

10. Under what limitation is the proposition in Example 30 of Chapter X. true for the hyperbola?

11. The ratio of the sines of the angles made by a diameter of an hyperbola with the asymptotes is equal to the ratio of the sines of the angles made by the conjugate diameter.

12. With two conjugate diameters of an ellipse as asymptotes a pair of conjugate hyperbolas is constructed; prove that if one hyperbola touch the ellipse the other will do so likewise; prove also that the diameters drawn through the points of contact are conjugate to each other.

CHAPTER XIII.

GENERAL EQUATION OF THE SECOND DEGREE.

269. WE shall now shew that every locus represented by an equation of the second degree is one of those which we have already discussed, that is, is one of the following; a point, a straight line, two straight lines, a circle, a parabola, an ellipse, or an hyperbola.

The general equation of the second degree may be written

$$ax^2 + bxy + cy^2 + dx + ey + f = 0;$$

we shall suppose the axes rectangular; if the axes were oblique we might transform the equation to one referred to rectangular axes, and as such a transformation cannot affect the degree of the equation (Art. 87), the transformed equation will still be of the form given above.

If the curve passes through the origin $f=0$; if the curve does *not* pass through the origin f is not $=0$, we may therefore divide by f and thus the equation will take the form

$$a'x^2 + b'xy + c'y^2 + d'x + e'y + 1 = 0.$$

270. We shall first investigate the possibility of removing from the equation the terms involving the *first* power of the variables.

Transfer the origin of co-ordinates to the point (h, k) by putting

$$x = x' + h, \quad y = y' + k,$$

and substituting these values of x and y in the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \dots\dots\dots\dots(1);$$

thus we obtain

$$ax'^2 + bx'y' + cy'^2 + (2ah + bk + d)\,x' + (2ck + bh + e)\,y' + f' = 0 \;\ldots\ldots\ldots (2),$$

where $$f' = ah^2 + bhk + ck^2 + dh + ek + f \;\ldots\ldots\ldots\ldots (3).$$

Now, if possible, let such values be assigned to h and k as will make the coefficients of x' and y' vanish; that is, let

$$2ah + bk + d = 0, \quad \text{and} \quad 2ck + bh + e = 0;$$

thus $$h = \frac{2cd - be}{b^2 - 4ac}, \quad k = \frac{2ae - bd}{b^2 - 4ac}.$$

It will therefore be possible to assign suitable values to h and k, *provided* $b^2 - 4ac$ *be not* $= 0$.

We shall see that the loci represented by the general equation of the second degree may be separated into two classes, those which have a *centre*, and those which in general have *not* a centre, and that in the former case $b^2 - 4ac$ is not zero, and in the latter case it is zero. We shall first consider the case in which $b^2 - 4ac$ is not zero, and consequently the values found above for h and k are finite.

Equation (2) thus becomes

$$ax'^2 + bx'y' + cy'^2 + f' = 0 \;\ldots\ldots\ldots\ldots\ldots (4).$$

Now if (4) is satisfied by any values x_1, y_1 of the variables, it is also satisfied by the values $-x_1$, $-y_1$. Hence the new origin of co-ordinates is the *centre* of the locus represented by (1).

Thus if $b^2 - 4ac$ be not $= 0$, the locus represented by (1) *has a centre*, and its co-ordinates are h and k, the values of which are given above.

The value of f' may be found by substituting the values of h and k in (3); the process may be facilitated thus; we have

$$2ah + bk + d = 0,$$

$$2ck + bh + e = 0.$$

Multiply the first of these equations by h, and the second by k, and add; thus

$$2ah^2 + 2ck^2 + 2bkh + dh + ek = 0,$$

or
$$2f' - dh - ek - 2f = 0;$$

$$\therefore f' = f + \frac{dh + ek}{2}$$

$$= f + \frac{cd^2 + ae^2 - bed}{b^2 - 4ac}.$$

We shall retain f' for shortness.

271. We may suppress the accents on the variables in equation (4) of the preceding article and write it

$$ax^2 + bxy + cy^2 + f' = 0 \text{.................} (5).$$

This equation we shall further simplify by changing the directions of the axes. (Art. 81.)

Put
$$x = x' \cos\theta - y' \sin\theta,$$
$$y = x' \sin\theta + y' \cos\theta,$$

and substitute in (5); thus

$$x'^2 (a \cos^2\theta + c \sin^2\theta + b \sin\theta \cos\theta)$$
$$+ y'^2 (a \sin^2\theta + c \cos^2\theta - b \sin\theta \cos\theta)$$
$$+ x'y' \{2(c-a) \sin\theta \cos\theta + b(\cos^2\theta - \sin^2\theta)\} + f' = 0 \text{...} (6).$$

Equate the coefficient of $x'y'$ to zero; thus

$$2(c-a) \sin\theta \cos\theta + b(\cos^2\theta - \sin^2\theta) = 0,$$

or
$$(c-a) \sin 2\theta + b \cos 2\theta = 0;$$

$$\therefore \tan 2\theta = \frac{b}{a-c} \text{ ... } (7).$$

Since θ can always be found so as to satisfy (7), the term involving $x'y'$ can be removed from (6), and the equation becomes

$$x'^2 (a \cos^2 \theta + c \sin^2 \theta + b \sin \theta \cos \theta)$$
$$+ y'^2 (a \sin^2 \theta + c \cos^2 \theta - b \sin \theta \cos \theta) + f' = 0,$$

or
$$Ax'^2 + By'^2 + f' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

where
$$A = \tfrac{1}{2} \{a + c + (a - c) \cos 2\theta + b \sin 2\theta\},$$
$$B = \tfrac{1}{2} \{a + c - (a - c) \cos 2\theta - b \sin 2\theta\}.$$

Since
$$\tan 2\theta = \frac{b}{a - c},$$

$$\cos 2\theta = \frac{\quad}{\sqrt{\{b^2 + (a - c)^2\}}},$$

and
$$\sin 2\theta = \frac{b}{\sqrt{\{b^2 + (a - c)^2\}}}.$$

Hence
$$A = \tfrac{1}{2} [a + c + \sqrt{\{b^2 + (a - c)^2\}}],$$
$$B = \tfrac{1}{2} [a + c - \sqrt{\{b^2 + (a - c)^2\}}].$$

We may suppress the accents on the variables in (8) and write it
$$-\frac{A}{f'} x^2 - \frac{B}{f'} y^2 = 1.$$

(1) If A, B, and f' have the same sign, the locus is impossible.

(2) If A and B have the same sign and f' have the contrary sign, the locus is an ellipse of which the semi-axes are respectively
$$\sqrt{\left(-\frac{f'}{A}\right)}, \text{ and } \sqrt{\left(-\frac{f'}{B}\right)}. \quad \text{(Art. 160.)}$$

The locus is of course a circle if $A = B$.

(3) If A and B have different signs, the locus is an hyperbola. (Art. 211.)

We have supposed in these three cases that f' is not $= 0$; if $f' = 0$, and A and B have the *same* sign the locus is the

origin; if $f'=0$, and A and B have *different* signs the locus consists of two straight lines represented by

$$y = \pm\sqrt{\left(-\frac{A}{B}\right)}\,x.$$

From the values of A and B we see that

$$AB = \frac{(a+c)^2 - b^2 - (a-c)^2}{4ac - b^2}.$$

Hence A and B have the same sign or different signs according as $b^2 - 4ac$ is negative or positive.

272. Hence we have the following summary of the results of the preceding articles of this chapter. The equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

represents an ellipse if $b^2 - 4ac$ be *negative*, subject to three exceptions in which it represents respectively a circle, a point, and an impossible locus. If $b^2 - 4ac$ be *positive*, the equation represents an hyperbola subject to one exception when it represents two intersecting straight lines.

273. We may notice that the equation in Art. 271,

$$\tan 2\theta = \frac{b}{a-c},$$

has an infinite number of solutions; for if 2α be *one* value of 2θ which satisfies the equation, then if $2\theta = 2\alpha + n\pi$, where n is any integer, the equation will be satisfied. But these different solutions will all give the same position for the axes. For the values of θ are comprised in the expression $\alpha + \frac{n\pi}{2}$, and by ascribing different values to n we obtain a series of angles each differing from α by a multiple of $\frac{\pi}{2}$, and the only changes that will arise from selecting different values of n are that the axis of x in one case may occupy the position of the axis of y

in another *and vice versâ*, or the positive and negative directions of the axes may be interchanged.

The radical in the value of $\cos 2\theta$ and of $\sin 2\theta$ in Art. 271 may have *either* sign; but the sign must be the same in both in order that the relation $\tan 2\theta = \frac{b}{a-c}$ may hold.

274. It appears from the former part of Art. 271, that by turning the axes through an angle θ the equation

$$ax^2 + bxy + cy^2 + f' = 0$$

becomes

$$a'x'^2 + b'x'y' + c'y'^2 + f' = 0,$$

where

$$a' = \tfrac{1}{2}\{a + c + (a-c)\cos 2\theta + b\sin 2\theta\},$$

$$b' = (c-a)\sin 2\theta + b\cos 2\theta,$$

$$c' = \tfrac{1}{2}\{a + c - (a-c)\cos 2\theta - b\sin 2\theta\}.$$

Hence $\qquad a' + c' = a + c;$ and

$$\begin{aligned} b'^2 - 4a'c' &= \{(c-a)\sin 2\theta + b\cos 2\theta\}^2 \\ &\qquad - (a+c)^2 + \{(a-c)\cos 2\theta + b\sin 2\theta\}^2 \\ &= (a-c)^2 + b^2 - (a+c)^2 \\ &= b^2 - 4ac. \end{aligned}$$

Thus the expression $b^2 - 4ac$ has the same value whether it be formed from the coefficients of the general equation of the second degree *before* or *after* the axes have been shifted.

The same remark applies to the expression $a + c$.

Hence we conclude that if the curve represented by the general equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

be a rectangular hyperbola, $a + c = 0$; for if the curve were referred to its transverse and conjugate diameters as axes this relation would hold, and therefore, as we have just seen, it must always hold whatever be the axes.

275. We have next to consider the case in which

$$b^2 - 4ac = 0.$$

We cannot now as in Art. 270 remove the terms involving the first power of the variables from the general equation, but we can still simplify the equation as in Art. 271, by changing the direction of the axes.

Let the equation be

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \text{ .... } (1);$$

put
$$x = x' \cos\theta - y' \sin\theta,$$
$$y = x' \sin\theta + y' \cos\theta,$$

then (1) becomes

$$x'^2 (a \cos^2\theta + c \sin^2\theta + b \sin\theta \cos\theta)$$
$$+ y'^2 (a \sin^2\theta + c \cos^2\theta - b \sin\theta \cos\theta)$$
$$+ x'y' \{2 (c - a) \sin\theta \cos\theta + b (\cos^2\theta - \sin^2\theta)\}$$
$$+ x' (d \cos\theta + e \sin\theta) + y' (e \cos\theta - d \sin\theta) + f = 0 \text{ .... } (2).$$

Now let
$$\tan 2\theta = \frac{b}{a - c},$$

then the coefficient of $x'y'$ in (2) vanishes, and as in Art. 271 the coefficients of x'^2 and y'^2 are

$$\tfrac{1}{2} [a + c \pm \sqrt{\{(a - c)^2 + b^2\}}].$$

One of these coefficients must therefore vanish since their product is $\frac{4ac - b^2}{4}$, which, by hypothesis, $= 0$; suppose the coefficient of $x'^2 = 0$, thus, by suppressing accents on the variables, (2) may be written

$$Cy^2 + Dx + Ey + f = 0 \text{ .... } (3).$$

If D be *not* $= 0$, this may be written

$$C\left(y + \frac{E}{2C}\right)^2 = -D\left(x - \frac{E^2}{4CD} + \frac{f}{D}\right),$$

and thus the locus is a parabola. (Art. 125.)

If $D = 0$, then (3) represents two parallel straight lines, or one straight line, or an impossible locus, according as E^2 is greater, equal to, or less than $4Cf$.

Hence if $b^2 - 4ac = 0$ the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

represents a parabola subject to three exceptions, in which it represents respectively two parallel straight lines, one straight line, and an impossible locus.

By combining this result with those enumerated in Art. 272, we have a complete account of the general equation of the second degree.

276. We have shewn in Art. 270, that when $b^2 - 4ac$ is not $= 0$, the general equation of the second degree represents a *central* curve; we shall now prove that when $b^2 - 4ac = 0$ the curve has *not* a centre *except when the locus consists of two parallel straight lines.*

If a curve of the second degree have the origin of co-ordinates for its centre, no term involving the first power of either of the variables alone can exist in the equation.

For if possible suppose that the origin of co-ordinates is the centre of the curve

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad\text{..............}\quad (1),$$

and let x_1, y_1 be the co-ordinates of a point on the curve, and therefore $-x_1$, $-y_1$ co-ordinates of another point on the curve; substitute successively in (1), then

$$ax_1^2 + bx_1y_1 + cy_1^2 + dx_1 + ey_1 + f = 0,$$

$$ax_1^2 + bx_1y_1 + cy_1^2 - dx_1 - ey_1 + f = 0;$$

therefore, by subtraction,

$$2(dx_1 + ey_1) = 0 \quad\text{......................}\quad (2).$$

Now unless d and e both vanish, (2) can only be true when (x_1, y_1) lies on the line

$$dx + ey = 0.$$

But the centre of a curve is a point which bisects *every* chord passing through it; hence the origin of co-ordinates

cannot be the centre of the curve (1) unless both d and e vanish.

277. Suppose then that we have an equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \text{......} \qquad (1),$$

in which $b^2 - 4ac = 0$. Here a and c cannot both be zero, for then b would also be zero, and (1) would not be an equation of the second degree; we shall suppose that a is not zero. Now if the curve denoted by (1) had a centre, and we took that centre as the origin of co-ordinates, the terms involving the first power of x and y would vanish by Art. 276. But from Arts. 270 and 274 it follows that when $b^2 - 4ac = 0$, we cannot *in general* make these terms vanish by changing the origin or the axes. The *only exception* that can arise is when the numerators in the values of h and k in Art. 270 vanish, so that the values of h and k become indeterminate, and the two equations for determining them reduce to one; see *Algebra*, Chapter XV. We have then $2ae - bd = 0$, so that $e = \frac{bd}{2a}$. Hence, by substituting for c and e, the equation (1) becomes

$$ax^2 + bxy + \frac{b^2y^2}{4a} + dx + \frac{bd}{2a}y + f = 0,$$

that is,
$$\left(x + \frac{by}{2a}\right)^2 + d\left(x + \frac{by}{2a}\right) + f = 0 \text{....} \qquad (2).$$

Equation (2) will furnish two values of $x + \frac{by}{2a}$, so that if these values are possible the locus consists of two parallel straight lines. In this case any point on the line which is parallel to these two and midway between them will be a centre.

Thus the result enunciated in the beginning of Art. 276 is proved.

278. We may observe that relations similar to those obtained in Art. 274 hold when the axes of co-ordinates are *oblique*. For suppose the equation

$$ax^2 + bxy + cy^2 + f' = 0$$

to be referred to rectangular axes, and let the axes be transformed into an oblique system inclined at an angle ω; suppose moreover that the new axis of x coincides with the old axis of x. We have then to put (Art. 84)

$$x = x' + y' \cos\omega, \quad y = y' \sin\omega;$$

substitute these values in the above equation and it becomes

$$a'x'^2 + b'x'y' + c'y'^2 + f' = 0,$$

where $\quad a' = a,$

$$b' = 2a\cos\omega + b\sin\omega,$$

$$c' = a\cos^2\omega + b\sin\omega\cos\omega + c\sin^2\omega;$$

thus $\quad b'^2 - 4a'c' = (b^2 - 4ac)\sin^2\omega,$

and $\quad a' + c' - b'\cos\omega = (a + c)\sin^2\omega;$

so that $\quad \dfrac{b'^2 - 4a'c'}{\sin^2\omega} = b^2 - 4ac,$

and $\quad \dfrac{a' + c' - b'\cos\omega}{\sin^2\omega} = a + c.$

Therefore, by means of Art. 274, we conclude that for any system of axes, rectangular or oblique, the expressions

$$\frac{b'^2 - 4a'c'}{\sin^2\omega} \text{ and } \frac{a' + c' - b'\cos\omega}{\sin^2\omega}$$

remain unchanged when the axes are changed.

See Salmon's *Conic Sections*, 3rd Edition, pages 142, 143.

279. We shall now shew how to trace a curve of the second degree from its equation without transformation of co-ordinates; the axes may be supposed oblique or rectangular.

Let the equation be

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \;\ldots\ldots\ldots\ldots\ldots\; (1).$$

Solve the equation with respect to y; thus

$$y = -\frac{bx+e}{2c} \pm \frac{1}{2c}\{(bx+e)^2 - 4c(ax^2+dx+f)\}^{\frac{1}{2}}$$

$$= -\frac{bx+e}{2c} \pm \frac{1}{2c}\{(b^2-4ac)x^2 + 2(be-2cd)x + e^2 - 4cf\}^{\frac{1}{2}} \ldots (2)$$

$$= ax + \beta \pm \left\{\frac{b^2-4ac}{4c^2}(x^2+2px+q)\right\}^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

where $\qquad \alpha = -\frac{}{2c}, \qquad \beta = -\frac{e}{2c},$

$$p = \frac{be-2cd}{b^2-4ac}, \quad q = \frac{e^2-4cf}{b^2-4ac}.$$

I. Suppose $b^2 - 4ac$ negative, and write $-\mu$ for $\frac{b^2-4ac}{4c^2}$; thus (3) becomes

$$y = ax + \beta \pm \{-\mu(x^2+2px+q)\}^{\frac{1}{2}} \ldots\ldots\ldots\ldots \quad (4).$$

Now $\quad x^2 + 2px + q = (x+p)^2 + q - p^2$;

if then $q - p^2$ be positive, the quantity under the radical is negative and the locus impossible;

if $q - p^2 = 0$, the locus is the point determined by

$$x = -p, \qquad y = ax + \beta;$$

if $q - p^2$ be negative, we may put

$$(x+p)^2 + q - p^2 = \{x + p + \sqrt{(p^2-q)}\}\{x + p - \sqrt{(p^2-q)}\}$$
$$= (x-\gamma)(x-\delta) \text{ suppose;}$$

and thus (4) may be written

$$y = ax + \beta \pm \{-\mu(x-\gamma)(x-\delta)\}^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots (5).$$

Since $(x-\gamma)(x-\delta)$ is positive, except when x lies between γ and δ, the values of y in (5) are real only so long as x lies between γ and δ. Moreover y is always *finite*, and thus the curve represented by (5) is limited in every direction.

Since we know from our previous investigations that (5) must represent one of the curves enumerated in Art. 269, it follows that it must represent an *ellipse*.

From the form of equation (5) we see that the chords parallel to the axis of y are bisected by the line

$$y = \alpha x + \beta \,. \qquad (6).$$

For let there be two points on the curve (5) having the common abscissa x_1, and the ordinates y', y'', respectively; and let y_1 be the corresponding ordinate of (6),

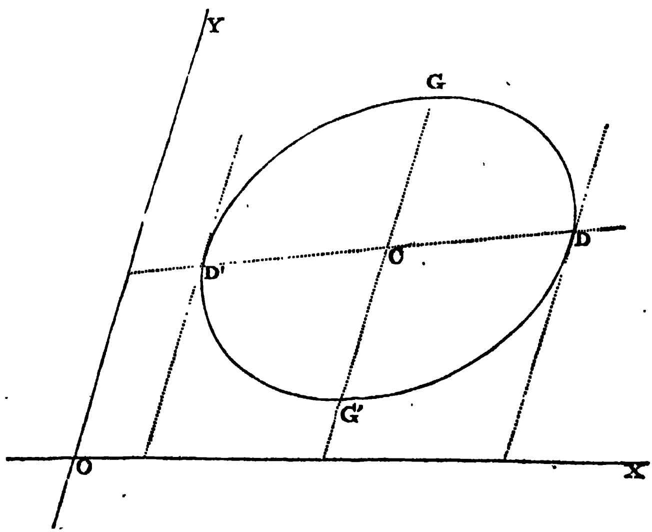
then $\quad y_1 = \alpha x_1 + \beta,$

$$y' = \alpha x_1 + \beta + \{-\mu (x_1 - \gamma)(x_1 - \delta)\}^{\frac{1}{2}},$$

$$y'' = \alpha x_1 + \beta - \{-\mu (x_1 - \gamma)(x_1 - \delta)\}^{\frac{1}{2}}.$$

Thus $\quad y_1 = \frac{1}{2}(y' + y'');$

and therefore the point (x_1, y_1) lies midway between the points (x_1, y') and (x_1, y'').



In the figure DCD' represents the line $y=\alpha x+\beta$; the abscissæ of D' and D are γ and δ respectively; supposing δ greater than γ. The centre C is midway between D' and D; its abscissa is therefore $\frac{1}{2}(\gamma+\delta)$. The equation to the curve will give the ordinates of D', D, G', G. Since GG' is parallel to the chords which $D'D$ bisects, DD' and GG' are conjugate diameters. GG' is a known quantity since the ordinates of G and G' are known. DD' is also a known quantity since the abscissæ and ordinates of D and D' are known. The angle between GG' and DD' is known from the equation to DD'; the axes of the ellipse may therefore be found (Arts. 193, 195).

II. Suppose b^2-4ac positive; put μ for $\frac{b^2-4ac}{4c^2}$; thus equation (3) becomes

$$y=\alpha x+\beta\pm\{\mu(x^2+2px+q)\}^{\frac{1}{2}} \quad\text{............}\quad (7).$$

Now $$x^2+2px+q=(x+p)^2+q-p^2;$$

if then $q-p^2$ be positive, the quantity under the radical is always positive, whatever positive or negative value be assigned to x. The curve therefore extends to infinity. Also it may be shewn as before, that the line

$$y=\alpha x+\beta$$

is a diameter of the curve; but it never *meets* the curve, because the quantity $x^2+2px+q$ or $(x+p)^2+q-p^2$ cannot vanish. Hence the curve consists of *two* unconnected branches extending to infinity, and is therefore an hyperbola.

If $q-p^2=0$, (7) becomes

$$y=\alpha x+\beta\pm\sqrt{\mu}\,(x+p).$$

The locus now consists of two intersecting lines.

If $q-p^2$ be negative we may as before write (7) in the form

$$y=\alpha x+\beta\pm\{\mu(x-\gamma)(x-\delta)\}^{\frac{1}{2}}.$$

Hence x may have any value, positive or negative, except those between γ and δ; hence the curve consists of two

unconnected branches extending to infinity, and is therefore an hyperbola.

We shall be assisted in drawing an example of this case by ascertaining the position of the asymptotes.

The equation to the curve is

$$y = \alpha x + \beta \pm \{\mu (x^2 + 2px + q)\}^{\frac{1}{2}};$$

$$\therefore \; y = \alpha x + \beta \pm x\sqrt{\mu}\left(1 + \frac{2p}{x} + \frac{q}{x^2}\right)^{\frac{1}{2}}.$$

Expand by the Binomial Theorem; thus

$$y = \alpha x + \beta \pm x\sqrt{\mu}\left\{1 + \tfrac{1}{2}\left(\frac{2p}{x} + \frac{q}{x^2}\right) + \&c.\right\}$$

$$= \alpha x + \beta \pm \sqrt{\mu}\,(x + p) + \&c.$$

The terms included in the &c. involve negative powers of x, and may therefore be made as small as we please by sufficiently increasing x; hence from the nature of an asymptote the required equations to the asymptotes are

$$y = \alpha x + \beta + \sqrt{\mu}\,(x + p),$$

and

$$y = \alpha x + \beta - \sqrt{\mu}\,(x + p).$$

Hence we can draw the asymptotes, and therefore the axes, for they bisect the angles between the asymptotes. The intersection of the asymptotes is the centre, and thus the situation and form of the hyperbola are known.

The expression

$$q - p^2 = \frac{(e^2 - 4cf)(b^2 - 4ac) - (be - 2cd)^2}{(b^2 - 4ac)^2};$$

this vanishes when

$$(e^2 - 4cf)(b^2 - 4ac) - (be - 2cd)^2 = 0,$$

and therefore when

$$(b^2 - 4ac)f + ae^2 + cd^2 - bed = 0;$$

so that if this relation holds the locus represented by (1) consists of two intersecting straight lines.

We have hitherto supposed that c is not zero, and as $b^2 - 4ac$ cannot be *negative* if c be zero, it was not necessary to advert to the possibility of c being zero while considering the first case. But as c may be zero consistently with $b^2 - 4ac$ being *positive*, we must now examine the consequences of supposing c zero.

The equation (1) may be solved with respect to x instead of with respect to y. Hence it will be found on investigation that the results hitherto obtained, when $b^2 - 4ac$ is positive, are certainly true provided that a and c are not *both* zero; the latter case requires further examination. Suppose then $a = 0$ and $c = 0$; thus (1) becomes

$$bxy + dx + ey + f = 0;$$

by changing the origin this can be put in the form

$$bx'y' + f' = 0,$$

where
$$f' = \frac{bf - de}{b};$$

the curve is therefore an hyperbola with the new axes for its asymptotes, except when $bf - de = 0$, and then it becomes two intersecting straight lines. When $a = 0$ and $c = 0$, the expression

$$(b^2 - 4ac) f + ae^2 + cd^2 - bed$$

reduces to $b(bf - de)$; thus we conclude that when $b^2 - 4ac$ is positive the equation (1) always represents an hyperbola, except when

$$(b^2 - 4ac) f + ae^2 + cd^2 - bed = 0,$$

and then it represents two intersecting straight lines.

III. Suppose $b^2 - 4ac = 0$, then (2) becomes

$$y = -\frac{bx + e}{2c} \pm \frac{1}{2c}\{2(be - 2cd)x + e^2 - 4cf\}^{\frac{1}{2}},$$

which may be written

$$y = \alpha x + \beta \pm \frac{1}{2c}(p'x + q')^{\frac{1}{2}},$$

where

$$\alpha = -\frac{b}{2c}, \quad \beta = -\frac{e}{2c},$$

$$p' = 2(be - 2cd), \quad q' = e^2 - 4cf.$$

If p' be positive, the expression under the radical is positive or negative, according as x is algebraically greater or less than $-\frac{q'}{p'}$; if p' be negative, the statement must be reversed. In both cases the curve extends to infinity in *one* direction only and is therefore a *parabola*.

The line $y = \alpha x + \beta$ is a diameter, bisecting all ordinates parallel to the axis of y, and meeting the parabola at the point for which $x = -\frac{q'}{p'}$.

If $p' = 0$, the equation becomes

$$y = \alpha x + \beta \pm \frac{\sqrt{q'}}{2c};$$

this equation represents two parallel straight lines if q' is positive, and one straight line if $q' = 0$; if q' is negative, the locus is impossible.

We have hitherto supposed in considering the third case that c is not zero; if $c = 0$, then $b = 0$, since $b^2 - 4ac = 0$; hence a and c cannot *both* be zero, for the equation (1) is supposed to be of the *second* degree. As before, we may solve equation (1) with respect to x, and thus determine the peculiarities which occur when $c = 0$. We have found for example when c is not zero, that the locus will consist of two parallel straight lines, when

$$be - 2cd = 0, \text{ and } e^2 - 4cf \text{ is positive;}$$

in like manner, if a be not zero, we can shew that the locus will consist of two parallel straight lines when

$$bd - 2ae = 0, \text{ and } d^2 - 4af \text{ is positive.}$$

By means of the relation $b^2 - 4ac = 0$, it is easily shewn that the second form of the conditions coincides with the first when a and c are both different from zero. When $a = 0$ the first is the necessary form of the conditions, but we see that the second form will then also hold. When $c = 0$ the second is the necessary form, though the first will then also hold. Hence we shall include every case by stating that *both* forms of the conditions must hold.

Similarly the conditions under which the locus will consist of one straight line, or will be impossible, may be investigated.

280. We will recapitulate the results of the present chapter with respect to the locus of the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

I. If $b^2 - 4ac$ be negative, the locus is an ellipse admitting of the following varieties:

(1) $c = a$, and $\dfrac{b}{2a} =$ cosine of the angle between the axes; locus a circle (Art. 104). coeff of x^2 = coeff. y^2.

(2) $(e^2 - 4cf)(b^2 - 4ac) - (be - 2cd)^2$ positive; locus impossible.

(3) $(e^2 - 4cf)(b^2 - 4ac) - (be - 2cd)^2 = 0$; locus a point.

II. If $b^2 - 4ac$ be positive, the locus is an hyperbola, except when

$$(b^2 - 4ac) f + ae^2 + cd^2 - bde = 0,$$

and then it consists of two intersecting straight lines.

III. If $b^2 - 4ac = 0$, the locus is a parabola, except when $be - 2cd = 0$, and $bd - 2ae = 0$; and then it consists of two parallel straight lines, or of one straight line, or is impossible, according as $e^2 - 4cf$ and $d^2 - 4af$ are positive, zero, or negative.

EXAMPLES.

1. Find the centre of the curve

$$x^2 - 4xy + 4y^2 - 2ax + 4ay = 0.$$

2. Find the centre of the ellipse

$$by\left(1 - \frac{y}{c}\right) + cx\left(1 - \frac{x}{b}\right) = xy.$$

3. What is represented by $ax^2 + 2bxy + cy^2 = 1$, when $b^2 = ac$?

4. Find the locus of the centre of a circle inscribed in a sector of a given circle, one of the bounding radii of the sector remaining fixed.

5. In the side AB of a triangle ABC, any point P is taken, and PQ is drawn perpendicular to AC; find the locus of the point of intersection of the straight lines BQ and CP.

6. DE is any chord parallel to the major axis AA' of an ellipse whose centre is C; and AD and CE intersect in P; shew that the locus of P is an hyperbola, and find the direction of its asymptotes.

7. Tangents to two concentric ellipses, the directions of whose axes coincide, are drawn from a point P, and the chords of contact intersect in Q; if the point P always lies on a straight line, shew that the locus of Q will be a rectangular hyperbola.

8. What form does the result in the preceding example take when two of the axes whose directions are coincident are equal?

9. Prove that an hyperbola may be described by the intersection of two straight lines which move parallel to themselves while the product of their distances from a fixed point remains constant.

10. Two lines are drawn from the focus of an ellipse including a constant angle; tangents are drawn to the ellipse at the points where the lines meet the ellipse; find the locus of the intersection of the tangents.

11. Find the latus rectum of the parabola

$$(y-x)^2 = ax.$$

12. Shew that the product of the semi-axes of the ellipse $y^2 - 4xy + 5x^2 = 2$ is 2.

13. Find the angle between the asymptotes of the hyperbola $xy = bx^2 + c$.

14. Find the equation to a parabola which touches the axis of x at a distance a, and cuts the axis of y at distances β, β' from the origin.

15. If two points be taken in each of two rectangular axes, so as to satisfy the condition that a rectangular hyperbola may pass through all the four, shew that the position of the hyperbola is indeterminate, and that its centre describes a circle which passes through the origin and bisects all the lines which join the points two and two.

16. Two lines of given lengths coincide with and move along two fixed axes in such a manner that a circle may always be drawn through their extremities; find the locus of the centre of the circle, and shew that it is an equilateral hyperbola.

17. A variable ellipse always touches a given ellipse, and has a common focus with it; find the locus of its other focus, (1) when the major axis is given, (2) when the minor axis is given.

18. Draw the curve

$$y^2 - 5xy + 6x^2 - 14x + 5y + 4 = 0.$$

19. Draw the curve

$$x^2 + y^2 - 3(x+y) - xy = 0.$$

20. Find the nature and position of the curve

$$y^2 - 8xy + 25x^2 + 6cy - 42cx + 9c^2 = 0.$$

21. The equation to a conic section being

$$ax^2 + 2bxy + cy^2 = 1,$$

shew that the equation to its axes is

$$xy(a-c) = b(x^2 - y^2).$$

22. The locus of the vertices of all similar triangles whose bases are parallel chords of a parabola will in general be another parabola; but if any one of the triangles *touch* the parabola with its sides, the locus becomes a straight line.

23. A series of circles pass through a given point O, have their centres in a line OA, and meet another line BC. Let M be the point in which one of the circles meets the line OA again, and let N be either of the points in which this circle meets BC. From M and N lines are drawn parallel to BC and OA respectively, intersecting in P; shew that the locus of P is an hyperbola which becomes a parabola when the two lines are at right angles.

24. The chord of contact of two tangents to a parabola subtends an angle β at the vertex; shew that the locus of their point of intersection is an hyperbola whose asymptotes are inclined to the axis of the parabola at an angle ϕ such that

$$\tan\phi = \tfrac{1}{2}\tan\beta.$$

25. Determine the locus of the middle points of the chords of the curve

$$ax^2 + 2bxy + cy^2 + 2ex + 2fy + g = 0,$$

which are parallel to the line $x\sin\theta - y\cos\theta = 0$; and hence find the position of the principal axes of the curve.

26. Shew that the equation

$$(x^2 - a^2)^2 + (y^2 - a^2)^2 = a^4$$

represents two ellipses.

CHAPTER XIV.

MISCELLANEOUS PROPOSITIONS.

281. WE shall give in this chapter some miscellaneous propositions for the most part applicable to all the conic sections.

To find the equation to a conic section, the origin and axes being unrestricted in position.

Let a, b be the co-ordinates of the focus; and let the equation to the directrix be

$$Ax + By + C = 0.$$

The distance of any point (x, y) from the focus is

$$\{(x-a)^2 + (y-b)^2\}^{\frac{1}{2}},$$

and the distance of the same point from the directrix is

$$\frac{Ax + By + C}{\sqrt{(A^2 + B^2)}}.$$

Let e be the *excentricity* of the conic section; then if (x, y) be a point on the curve, we have, by definition,

$$\{(x-a)^2 + (y-b)^2\}^{\frac{1}{2}} = \frac{e\,(Ax + By + C)}{\sqrt{(A^2 + B^2)}} \text{.........(1)};$$

$$\therefore\ (x-a)^2 + (y-b)^2 = \frac{e^2\,(Ax + By + C)^2}{A^2 + B^2} \text{.........(2)}.$$

We see from (1) that the distance of any point on a conic section from the focus can be expressed in terms of the *first*

power of the co-ordinates of that point whatever be the origin and axes. This is usually expressed by saying *the distance of any point from the focus is a linear function of the co-ordinates of the point.*

282. It will be seen by examining the equations to the conic sections given in the preceding chapters that any conic section may be represented by the equation

$$y^2 = mx + nx^2.$$

$y^2 = mx - nx^2$

The origin is a vertex of the curve and the axis of x an axis of the curve; m is the latus rectum; in the parabola $n = 0$; n is negative in the ellipse and positive in the hyperbola. In the circle m is the diameter of the circle and $n = -1$.

283. *To find the equation to the tangent at any point of a curve of the second degree.*

Let the equation to the curve be

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \text{ ....(1)},$$

the axes being oblique or rectangular.

Let x', y' be the co-ordinates of the point,

x'', y'' the co-ordinates of an adjacent point on the curve.

The equation to the secant through these points is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \text{ ....(2)}.$$

Since (x', y') and (x'', y'') are on the curve,

$$ax'^2 + bx'y' + cy'^2 + dx' + ey' + f = 0,$$

$$ax''^2 + bx''y'' + cy''^2 + dx'' + ey'' + f = 0;$$

$$\therefore\ a(x''^2 - x'^2) + b(x''y'' - x'y') + c(y''^2 - y'^2)$$
$$+ d(x'' - x') + e(y'' - y') = 0,$$

or $(x''-x')\{a(x''+x')+by''+d\}$

$$+(y''-y')\{c(y''+y')+bx'+e\}=0;$$

$$\therefore \frac{y''-y'}{x''-x'}=-\frac{a(x''+x')+by''+d}{c(y''+y')+bx'+e}.$$

Hence (2) may be written

$$y-y'=-\frac{a(x''+x')+by''+d}{c(y''+y')+bx'+e}(x-x').$$

Now in the limit $x''=x'$ and $y''=y'$; hence the equation to the tangent at the point (x', y') is

$$y-y'=-\frac{2ax'+by'+d}{2cy'+bx'+e}(x-x')\text{............}(3).$$

This equation may be simplified; we have by reduction

$$\begin{aligned}y(2cy'+bx'+e)&+x(2ax'+by'+d)\\&=y'(2cy'+bx'+e)+x'(2ax'+by'+d)\\&=2(ax'^2+bx'y'+cy'^2+dx'+ey'+f)-dx'-ey'-2f;\end{aligned}$$

$$\therefore y(2cy'+bx'+e)+x(2ax'+by'+d)+dx'+ey'+2f=0,$$
$$\text{.........}(4).$$

If $f=0$, the curve passes through the origin, and the equation to the tangent at that point becomes

$$y=-\frac{d}{e}x,$$

which we see does not involve the coefficients of x^2, y^2, or xy, in the equation to the curve.

284. The equation to the normal at the point (x', y') when the curve is expressed by equation (1) of the preceding article and the axes are rectangular, will be

$$y-y'=\frac{2cy'+bx'+e}{2ax'+by'+d}(x-x').$$

285. It may be shewn as in Art. 183, that if from a point (h, k) two tangents be drawn to the curve expressed by equation (1) of Art. 283, the equation to the *chord of contact* is

$$y(2ck + bh + e) + x(2ah + bk + d) + dh + ek + 2f = 0.$$

286. *All chords of a conic section which subtend a right angle at a given point of the curve intersect in the normal at that point.*

Take the given point of the curve as the origin of a system of rectangular axes, and let the equation to the curve be

$$ax^2 + bxy + cy^2 + dx + ey = 0 \ldots\ldots\ldots\ldots (1).$$

The axis of x meets the curve at the points found by making $y = 0$ in the above equation, that is, at the points $x = 0$, and $x = -\dfrac{d}{a}$.

Similarly the axis of y meets the curve at the origin and also at the point for which $y = -\dfrac{e}{c}$.

Hence the equation

$$\frac{x}{-\dfrac{d}{a}} + \frac{y}{-\dfrac{e}{c}} = 1,$$

or

$$\frac{xa}{d} + \frac{yc}{e} + 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

represents the chord joining the points of intersection of the axes and curve.

Also the equation to the normal to the curve at the origin is by Art. 284,

$$y = \frac{e}{d}x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Hence (2) and (3) meet in the point whose co-ordinates are

$$\frac{-d}{a+c}, \quad \frac{-e}{a+c},$$

and whose distance from the origin is therefore

$$\frac{\sqrt{(d^2+e^2)}}{a+c}.$$

Now change the *directions* of the axes preserving the same origin; the equation (1) will then become

$$a'x'^2+b'x'y'+c'y'^2+d'x'+e'y'=0.$$

Also it appears from Arts. 274 and 275, that

$$a'+c'=a+c, \text{ and } d'^2+e'^2=d^2+e^2.$$

Hence the normal at the origin will meet the new chord at the same distance from the origin as it met the original chord, that is, will meet it *in the same point*. Since this is true whatever be the directions of the axes, it follows that all the chords intersect in the same point.

287. By comparing Arts. 154, 204, and 264, we see that the polar equation to any conic section, the focus being the pole and the initial line the axis, is

$$r=\frac{l}{1+e\cos\theta},$$

where l = half the latus rectum.

We shall use this in proving the following proposition.

The semi-latus rectum of any conic section is an harmonic mean between the segments made by the focus of any focal chord of that conic section.

Let $A'SP=\theta$, see fig. to Art. 158;

$$\therefore\ SP=\frac{l}{1+e\cos\theta}.$$

Suppose PS produced to meet the curve again in P';

$$SP'=\frac{l}{1+e\cos(\pi+\theta)};$$

$$\therefore \frac{1}{SP}+\frac{1}{SP'}=\frac{1+e\cos\theta}{l}+\frac{1-e\cos\theta}{l}$$

$$=\frac{2}{l},$$

which proves the proposition.

288. The polar equation to the tangent to a conic section, the focus being the pole and the initial line the axis, is (Art. 205)

$$\frac{l}{r}=e\cos\theta+\cos(\alpha-\theta) \;\ldots\ldots\ldots\ldots\ldots\; (1),$$

where α is the angular co-ordinate of the point of contact.

Similarly the polar equation to the tangent at the point whose angular co-ordinate is β, is

$$\frac{l}{r}=e\cos\theta+\cos(\beta-\theta) \;\ldots\ldots\ldots\ldots\ldots\; (2).$$

At the point where these tangents meet, we have

$$\cos(\alpha-\theta)=\cos(\beta-\theta).$$

Now we cannot have

$$\alpha-\theta=\beta-\theta,$$

since α and β are by supposition different; we therefore take

$$\alpha-\theta=\theta-\beta,$$

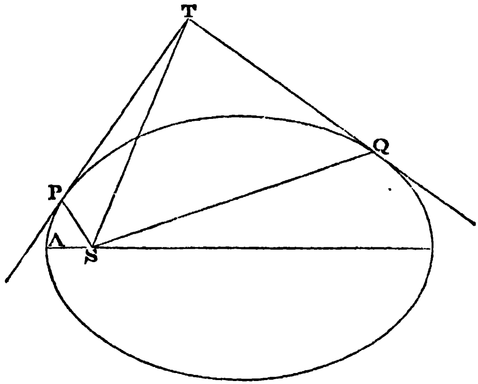
$$\therefore\ \theta=\frac{\alpha+\beta}{2}.$$

Thus the two tangents (1) and (2) meet at the point whose angular co-ordinate is $\frac{\alpha+\beta}{2}$.

For example, suppose the conic section an ellipse; let

$$ASP=\alpha,\quad ASQ=\beta,$$

and let the tangents at P and Q meet at T;



then $$AST = \frac{\alpha + \beta}{2};$$

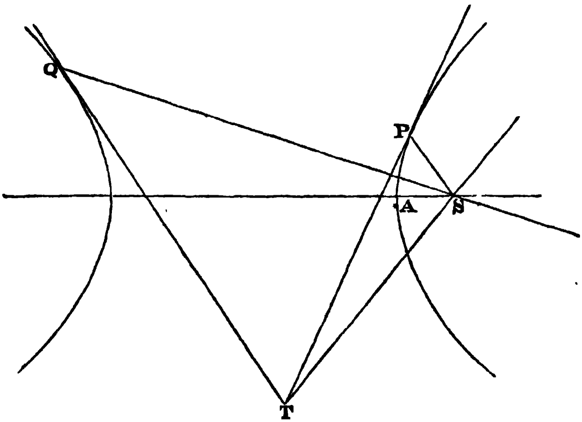
$$\therefore\ PST = \frac{\beta - \alpha}{2} = QST;$$

that is, the *two tangents drawn from any point to an ellipse subtend equal angles at either focus.*

Similarly the two tangents drawn from any point to a parabola subtend equal angles at the focus.

With respect to the hyperbola we have to distinguish two cases. We have shewn in Art. 231, that from any point included between the asymptotes and the curve, two tangents can be drawn both meeting the *same* branch of the curve, but from any point included within the supplemental angles of the asymptotes two tangents can be drawn meeting *different* branches of the curve.

If now the two tangents from a point meet the *same* branch of an hyperbola, it may be shewn as in the case of the ellipse, that they subtend equal angles at either focus. We will consider the case in which the tangents meet *different* branches.



Let T be a point from which tangents TP, TQ are drawn to different branches of an hyperbola.

Let $ASP = \alpha$; and let the angle which QS produced through S makes with AS be β; then β is an angle greater than π, and $ASQ = \beta - \pi$.

Thus the equations to TP and TQ will be respectively

$$\frac{l}{r} = e\cos\theta + \cos(\alpha - \theta), \quad \frac{l}{r} = e\cos\theta + \cos(\beta - \theta).$$

At the point T where they meet, we have

$$\cos(\alpha - \theta) = \cos(\beta - \theta).$$

We may therefore take $\theta = \frac{\alpha+\beta}{2}$, that is, we have $\frac{\alpha+\beta}{2}$ as the angle which TS produced makes with AS; thus

$$AST = \pi - \frac{\alpha+\beta}{2}$$

$$\therefore\ TSP = \pi - \frac{\beta-\alpha}{2} \qquad TSQ = \frac{\beta-\alpha}{2}.$$

$$\therefore\ TSP + TSQ = \pi;$$

that is, the angle which one tangent subtends at either focus is the supplement of the angle which the other tangent subtends at the same focus.

289. We have given in Art. 120 the definitions of a pole and polar with respect to a given circle. The same definitions are used generally substituting *conic section* for *circle*. If then the equation to the curve be

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

the equation to the *polar* of (x', y') is (Art. 283)

$$x(2ax' + by' + d) + y(2cy' + bx' + e) + dx' + ey' + 2f = 0.$$

290. *If one straight line pass through the pole of another straight line, the second straight line will pass through the pole of the first straight line.*

Let (x', y') be the pole of the *first* straight line, and therefore

$$x(2ax' + by' + d) + y(2cy' + bx' + e) + dx' + ey' + 2f = 0 \ldots (1)$$

is the equation to the *first* straight line.

Let (x'', y'') be the pole of the *second* straight line, and therefore

$$x(2ax'' + by'' + d) + y(2cy'' + bx'' + e) + dx'' + ey'' + 2f = 0 \ldots\ldots\ldots (2)$$

is the equation to the *second* straight line.

Since (1) passes through (x'', y'') we have

$$x''(2ax' + by' + d) + y''(2cy' + bx' + e) + dx' + ey' + 2f = 0,$$

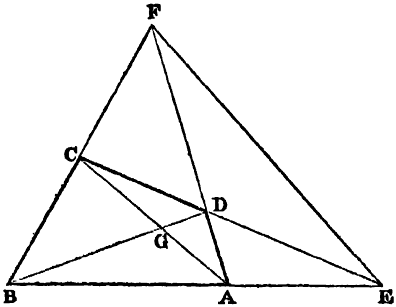
that is,

$$x'(2ax'' + by'' + d) + y'(2cy'' + bx'' + e) + dx'' + ey'' + 2f = 0;$$

hence (2) passes through (x', y').

291. *The intersection of two straight lines is the pole of the line which joins the poles of those lines.* See Art. 122.

292. *If a quadrilateral* ABCD *be inscribed in a conic section, of the three points* E, F, G, *each is the pole of the line joining the other two.*



Let E be the origin; EA, ED the directions of the axes of x and y; and let the equation to the conic section be

$$ax^2+bxy+cy^2+dx+ey+f=0 \qquad (1).$$

Also suppose

$$EA=h, \qquad EB=h',$$
$$ED=k, \qquad EC=k'.$$

The equation to AC is $\frac{x}{h}+\frac{y}{k'}=1$ .... (2),

.... BD ... $\frac{x}{h'}+\frac{y}{k}=1$ .... (3),

.... AD ... $\frac{x}{h}+\frac{y}{k}=1$ .... (4),

.... CB ... $\frac{x}{h'}+\frac{y}{k'}=1$ .... (5).

From (2) and (3) it follows that the equation

$$x\left(\frac{1}{h}+\frac{1}{h'}\right)+y\left(\frac{1}{k}+\frac{1}{k'}\right)=2. \qquad (6)$$

represents *some* line passing through G. But from (4) and (5) it follows that (6) represents *some* line passing through F. Hence (6) must be the equation to FG.

Suppose in (1) that $y=0$; then we have the quadratic

$$ax^2+dx+f=0;$$

and the roots of this equation are h and h'; hence

$$h+h'=-\frac{d}{a}, \qquad hh'=\frac{f}{a};$$

$$\frac{1}{h}+\frac{1}{h'}=-\frac{d}{f}.$$

Similarly, $$\frac{1}{k}+\frac{1}{k'}=-\frac{e}{f}.$$

Hence (6) becomes

$$dx+ey+2f=0.$$

But this, by Art. 289, is the equation to the polar of the origin; therefore FG is the polar of E. Similarly EG is the polar of F. Hence, by Art. 291, G is the pole of EF.

293. *To determine the form of the general equation to a conic section when the axes are tangents.*

Let $$ax^2+bxy+cy^2+dx+ey+f=0 \text{ .... } (1)$$
be the equation to the conic section.

To find where the curve meets the axis of x, put $y=0$ in the above equation; thus

$$ax^2+dx+f=0.$$

If the axis of x is a *tangent* to the curve it must meet the curve in only one point (see Art. 171); hence the roots of the above quadratic must be equal; therefore

$$d^2=4af \text{ .... } (2).$$

Similarly that the axis of y may be a tangent to (1) we must have

$$e^2=4cf \text{ .... } (3).$$

Substitute the values of a and c from (2) and (3), then (1) becomes

$$d^2x^2 + 4dfx + e^2y^2 + 4efy + 4bfxy + 4f^2 = 0,$$

or
$$(dx + ey + 2f)^2 + (4bf - 2de)\,xy = 0,$$

or
$$\left(\frac{d}{2f}x + \frac{e}{2f}y + 1\right)^2 + \frac{2bf - de}{2f^2}\,xy = 0.$$

Put
$$\frac{d}{2f} = -\frac{1}{h}, \qquad \frac{e}{2f} = -\frac{1}{k}, \qquad \frac{2bf - de}{2f^2} = \mu;$$

thus we obtain for the required equation

$$\left(\frac{x}{h} + \frac{y}{k} - 1\right)^2 + \mu xy = 0.$$

By putting successively x and $y = 0$, we see that h is the distance from the origin to the point where the curve meets the axis of x, and k is the distance from the origin to the point where the curve meets the axis of y.

If it be required to determine a conic section which touches two given straight lines in given points, and also passes through another given point, we may assume the last written equation to represent it, so that the lines to be touched are taken as the axes of x and y; then by putting the co-ordinates of the additional given point in the equation we find a single value for μ. Thus there is only one conic section satisfying the data.

294. Suppose the equation

$$\left(\frac{x}{h} + \frac{y}{k} - 1\right)^2 + \mu xy = 0 \ldots.. \qquad (1)$$

to represent a parabola. Then, by Art. 280,

$$\left(\frac{2}{hk} + \mu\right)^2 = \frac{4}{h^2k^2};$$

$$\therefore\ \mu = 0, \quad \text{or } \mu = -\frac{4}{hk}.$$

If $\mu = 0$, (1) becomes

$$\frac{x}{h} + \frac{y}{k} - 1 = 0;$$

this equation represents the straight line joining the points of contact of (1) with the axes.

If $\mu = -\frac{4}{hk}$, we have from (1),

$$\left(\frac{x}{h} + \frac{y}{k} - 1\right)^2 = \frac{4xy}{hk} \qquad .(2);$$

$$\therefore \frac{x}{h} + \frac{y}{k} - 1 = \pm 2$$

$$\therefore \frac{x}{h} \mp 2\sqrt{\left(\frac{xy}{hk}\right)} + \frac{y}{k} = 1;$$

$$\therefore \sqrt{\frac{x}{h}} \mp \sqrt{\frac{y}{k}} = \pm 1.$$

We may write this

$$\sqrt{\frac{x}{h}} + \sqrt{\frac{y}{k}} = 1 \qquad (3),$$

remembering that the radicals may be positive or negative. Thus (3) is the equation to a parabola referred to two tangents as axes.

295. We may notice the form of the equation to the tangent to the parabola

$$\sqrt{\frac{x}{h}} + \sqrt{\frac{y}{k}} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation to the secant through (x', y') and (x'', y'') is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x').$$

Since (x', y') and (x'', y'') are on the parabola, we have

$$\sqrt{\frac{x'}{h}}+\sqrt{\frac{y'}{k}}=1, \text{ and}$$

$$\sqrt{\frac{x''}{h}}+\sqrt{\frac{y''}{k}}=1;$$

$$\therefore \frac{\sqrt{x''}-\sqrt{x'}}{\sqrt{h}}=-\frac{\sqrt{y''}-\sqrt{y'}}{\sqrt{k}};$$

and $$\frac{y''-y'}{x''-x'}=\frac{\sqrt{y''}-\sqrt{y'}}{\sqrt{x''}-\sqrt{x'}}\cdot\frac{\sqrt{y''}+\sqrt{y'}}{\sqrt{x''}+\sqrt{x'}}=-\frac{\sqrt{k}}{\sqrt{h}}\cdot\frac{\sqrt{y''}+\sqrt{y'}}{\sqrt{x''}+\sqrt{x'}}.$$

Hence the equation to the secant may be written

$$y-y'=-\frac{\sqrt{k}}{\sqrt{h}}\cdot\frac{\sqrt{y''}+\sqrt{y'}}{\sqrt{x''}+\sqrt{x'}}(x-x').$$

Hence we have for the equation to the tangent at (x', y')

$$y-y'=-\frac{\sqrt{(ky')}}{\sqrt{(hx')}}(x-x'),$$

or $$\frac{y}{\sqrt{(ky')}}+\frac{x}{\sqrt{(hx')}}=\frac{y'}{\sqrt{(ky')}}+\frac{x'}{\sqrt{(hx')}}=1.$$

Similar Curves.

296. Def. Two curves are said to be *similar* and *similarly* situated when a radius vector drawn from some fixed point in any direction to the first curve bears a constant ratio to the radius vector drawn from some fixed point in a parallel direction to the second curve.

Two curves are said to be *similar* when a radius vector drawn from some fixed point in any direction to the first curve bears a constant ratio to the radius vector drawn from some fixed point to the second curve in a direction inclined at a constant angle to the former.

The two fixed points are called *centres of similarity.*

297. If two curves are similar, so that a pair of *centres of similarity* exists, then an infinite number of pairs of centres of similarity can be found.

For, suppose O, O' to denote one pair of *centres of similarity;* and let OP, OQ be radii vectores of the first curve, and $O'P'$, $O'Q'$ the corresponding radii vectores of the second curve, so that the angle $POQ =$ the angle $P'O'Q'$, and

$$\frac{OP}{O'P'} = \frac{OQ}{O'Q'}.$$

Suppose any point S taken and joined to O; then make the angle $P'O'S' =$ the angle POS, the angles being measured in the same direction, and take $O'S'$ so that

$$\frac{O'S'}{OS} = \frac{O'P'}{OP};$$

then S and S' shall be centres of similarity.

For join SP, SQ, $S'P'$, $S'Q'$; then the triangles SOP, $S'O'P'$ are similar; and so also are the triangles SOQ, $S'O'Q'$. Hence it easily follows that

$$\text{the angle } QSP = Q'S'P';$$

and
$$\frac{SP}{S'P'} = \frac{SQ}{S'Q'};$$

and thus the proposition is established.

298. *All parabolas are similar curves.*

Let $4a$ be the latus rectum of a parabola, and $4a'$ the latus rectum of a second parabola. The polar equations of these curves, the foci being the respective poles, are

$$r = \frac{2a}{1 + \cos\theta},$$

$$r' = \frac{2a'}{1 + \cos\theta'}.$$

Hence, if $\theta = \theta'$, we have

$$\frac{r}{r'} = \frac{a}{a'}.$$

Thus any two parabolas are similar, and the foci are centres of similarity.

299. To find the conditions that must hold that the curves

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \text{ .... } (1),$$

$$a'x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0 \text{ .... } (2),$$

may be similar and similarly situated.

Suppose (h, k), (h', k') the respective centres of similarity; for x and y in (1) put

$$h + r\cos\theta, \quad \text{and } k + r\sin\theta$$

respectively; we shall thus obtain a quadratic in r which may be written

$$Lr^2 + Mr + N = 0 \text{ .... } (3).$$

For x and y in (2) put

$$h' + r'\cos\theta, \quad \text{and } k' + r'\sin\theta$$

respectively; we shall thus obtain a quadratic in r' which may be written

$$L'r'^2 + M'r' + N' = 0 \text{ .... } (4).$$

Now that the curves may be similar and similarly situated, we must always have $r' = \lambda r$, where λ is some constant quantity; thus (4) becomes

$$\lambda^2 L'r^2 + \lambda M'r + N' = 0 \text{ .... } (5).$$

Since (3) or (5) will give the values of r, these equations must be *identical;* thus

$$\frac{L}{\lambda^2 L'} = \frac{M}{\lambda M'} = \frac{N}{N'} \text{ .... } (6).$$

Since neither N nor N' involves θ, we deduce as a necessary condition that $\frac{L}{L'}$ must be constant whatever θ may be. Put for L and L' their values; then

$$\frac{a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta}{a'\cos^2\theta + b'\sin\theta\cos\theta + c'\sin^2\theta} = \text{a constant} = \mu \text{ say....} (7);$$

$$\therefore\ (a - \mu a')\cos^2\theta + (b - \mu b')\sin\theta\cos\theta + (c - \mu c')\sin^2\theta = 0.$$

Since this is to be true whatever θ may be, it follows that

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} \dots\dots\dots\dots\dots\dots\dots \quad (8).$$

Hence we have arrived at (8) as *necessary* conditions, in order that (1) and (2) may be similar and similarly situated. We have still to ascertain whether these are *sufficient* to ensure the similarity. The direct method would be to examine if h, k, h', k' can be so chosen as to make (6) hold; but the following method is more simple. The equations (1) and (2), by means of (8), may be written

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$
$$ax^2 + bxy + cy^2 + \mu(d'x + e'y + f') = 0.$$

I. Suppose $b^2 - 4ac = 0$; then each curve is in general a parabola, and therefore the curves are similar; also their diameters are parallel so that the curves are similarly situated. See Art. 279. This conclusion is subject to the exceptions that may arise when either locus instead of a parabola, becomes one or two straight lines, or impossible.

II. Suppose $b^2 - 4ac$ not $= 0$. We may then by changing the origin of co-ordinates for each curve reduce the equations to the form

$$ax^2 + bxy + cy^2 + f_1 = 0,$$
$$ax^2 + bxy + cy^2 + f_2 = 0.$$

By expressing these equations in polar co-ordinates, they give

$$r^2 = \frac{-f_1}{a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta},$$

$$\frac{-f_2}{a\cos^2\theta' + b\sin\theta'\cos\theta' + c\sin^2\theta'}.$$

Thus, if $\theta = \theta'$, we have $\frac{r}{r'} =$ constant. Hence the curves are in general similar and similarly situated. This conclusion is subject to the exceptions that may arise when either locus instead of a curve becomes two straight lines, or a point, or impossible.

300. Next, suppose we require the curves (1) and (2) of Art. 299 to be similar *without the limitation of being similarly situated.* For x and y in (1) we put respectively

$$h + r\cos\theta, \qquad k + r\sin\theta.$$

For x and y in (2) we put respectively

$$h' + r'\cos(\theta+\alpha), \qquad k' + r'\sin(\theta+\alpha),$$

where α is some constant angle at present undetermined. Proceed as in Article 299; instead of equation (7) we shall now have

$$\frac{a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta}{a'\cos^2(\theta+\alpha) + b'\sin(\theta+\alpha)\cos(\theta+\alpha) + c'\sin^2(\theta+\alpha)}$$
$$= \text{a constant} = \mu \text{ say.}$$

This may be written

$$\frac{a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta}{A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta} = \mu,$$

where

$$A = a'\cos^2\alpha + c'\sin^2\alpha + b'\sin\alpha\cos\alpha,$$

$$B = 2(c' - a')\sin\alpha\cos\alpha + b'(\cos^2\alpha - \sin^2\alpha),$$

$$C = a'\sin^2\alpha + c'\cos^2\alpha - b'\sin\alpha\cos\alpha.$$

That the curves may be similar we must have

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c}.$$

Hence each of these ratios must equal $\dfrac{A+C}{a+c}$.

$$\therefore \frac{B^2}{b^2} = \frac{(A+C)^2}{(a+c)^2};$$

$$\therefore \frac{B^2}{(A+C)^2} = \frac{b^2}{(a+c)^2}.$$

And
$$\frac{AC}{ac} = \frac{(A+C)^2}{(a+c)^2};$$

$$\therefore \frac{AC}{(A+C)^2} = \frac{ac}{(a+c)^2}.$$

Hence, $$\frac{B^2-4AC}{(A+C)^2} = \frac{b^2-4ac}{(a+c)^2}.$$

But $$A+C=a'+c',$$

and $$B^2-4AC=b'^2-4a'c', \text{ (Art. 274)};$$

$$\therefore \frac{b'^2-4a'c'}{(a'+c')^2} = \frac{b^2-4ac}{(a+c)^2}.$$

This relation must therefore hold, in order that the given curves may be similar.

EXAMPLES.

1. Straight lines are drawn through a fixed point; shew that the locus of the middle points of the portions of them intercepted between two fixed straight lines is an hyperbola whose asymptotes are parallel to those fixed lines.

2. Through any point P of an ellipse QPQ' is drawn parallel to the axis major, and PQ and PQ' each made equal to the focal distance SP, find the loci of Q and Q'.

3. In the given right lines AP, AQ, are taken variable points p, q, such that $Ap : pP :: Qq : qA$; prove that the locus of the point of intersection of Pq and Qp is an ellipse which touches the given right lines in the points P, Q.

4. TP, TQ are two tangents to a parabola, P, Q being the points of contact; a third tangent cuts these in p, q respectively; shew that

$$\frac{Tp}{TP} + \frac{Tq}{TQ} = 1.$$

5. TP, TQ are *equal* tangents to a parabola, P, Q being the points of contact; if PT, QT be both cut by a third tangent, prove that their alternate segments will be equal.

6. From a point O are drawn two lines to touch a parabola in the points P and Q; another line touches the parabola in R and intersects OP, OQ, in S and T; if V be the intersection of the lines joining PT, QS, crosswise, O, R, V are in the same straight line.

7. From an external point two tangents are drawn to an ellipse; shew that an ellipse similar and similarly situated will pass through the external point, the points of contact, and the centre of the given ellipse.

8. A and B are two similar, similarly situated, and concentric ellipses; C is a third ellipse similar to A and B, its centre being on the circumference of B, and its axes parallel to those of A or B; shew that the chord of intersection of A and C is parallel to the tangent to B at the centre of C.

9. The line joining any point with the intersection of the polar of that point with a directrix subtends a right angle at the corresponding focus.

10. If normals be drawn to an ellipse from a given point, the points where they cut the curve will lie on a rectangular hyperbola which passes through the given point and has its asymptotes parallel to the axes of the ellipse.

11. If CM, MP are the abscissa and ordinate of any point P, on the circumference of a circle, and MQ is taken equal to MP and inclined to it at a constant angle, the locus of the point Q is an ellipse.

12. Having given the equation to a conic section

$$ax^2 + 2bxy + y^2 + f = 0,$$

find the locus of the intersection of normals drawn at the extremities of each pair of ordinates to the same abscissa.

13. Any two points P, Q are taken in two fixed lines in one plane such that the line PQ is always parallel to a given line; P, Q are severally joined with two fixed points H, R; find the locus of the intersection of PH and QR.

14. The tangent at any point P of a circle meets the tangent at a fixed point A in T, and T is joined with B the extremity of the diameter passing through A; shew that the locus of the point of intersection of AP and BT, is an ellipse.

6 15. The polar equation to a conic section from the focus being

$$\frac{1}{r} - c \cos \theta = b,$$

shew that the equation to a straight line which cuts it at the points for which $\theta = \alpha$ and β respectively, is

$$\frac{1}{r} - c \cos \theta = b \cos \left(\theta - \frac{\alpha + \beta}{2}\right) \sec \frac{\alpha - \beta}{2}.$$

16. Chords are drawn in a conic section so as to subtend a constant angle at the focus; prove that the locus of the foot of the perpendicular dropped from the focus upon the chord is a circle, except in a particular case when it becomes a straight line.

17. If SP, SQ be focal distances of a conic section including a constant angle; shew that PQ touches a confocal conic.

18. Having given two fixed points through which a conic section is to pass, and the directrix, find the locus of the corresponding focus.

19. The focus and directrix of an ellipse are given; through the former a line is drawn making with the latter an angle whose sine is the excentricity of the ellipse. Find the locus of the points where this line meets the curve, the excentricity being variable.

20. A series of conic sections is described having a common focus and directrix, and in each curve a point is taken whose distance from the focus varies inversely as the latus rectum; find the locus of these points.

21. Two conic sections have a common focus S through which any radius vector is drawn meeting the curves in P, Q, respectively. Prove that the locus of the point of intersection of the tangents at P, Q, is a straight line.

Shew that this straight line passes through the intersection of the directrices of the conic sections, and that the sines of the angles which it makes with these lines are inversely proportional to the corresponding excentricities.

22. A line is drawn cutting an ellipse in the points P, p; let Q be either of the points in which the same line meets a similar, similarly situated, and concentric ellipse; shew that if the line moves parallel to itself, $PQ \,.\, Qp$ is constant.

23. In two straight lines OX, OY, which intersect in O, take $OA = a$, $OB = b$; shew that the centres of all the conic sections which touch the lines in A and B lie on the straight line

$$ay = bx.$$

24. About two equal ellipses whose centres coincide, and whose major axes are inclined to each other at a given angle an ellipse is circumscribed; if A and B be the semi-axes of the circumscribing ellipse, a and b the semi-axes of the equal ellipses, and 2α the inclination of their major axes, then will

$$a^2b^2 + A^2B^2 = (A^2b^2 + B^2a^2)\cos^2\alpha + (A^2a^2 + B^2b^2)\sin^2\alpha.$$

Hence shew that about the two equal ellipses a *similar* ellipse may be circumscribed.

25. Two similar ellipses have a common centre and touch each other; if n be the ratio of their linear magnitudes, m the ratio of the major to the minor axis in either, and α the inclination of their major axes, prove that

$$\sin\alpha = \frac{n - \dfrac{1}{n}}{m - \dfrac{1}{m}}.$$

26. Two tangents (a, b) to a parabola intersect in P at an angle ω, and a circle is described between these tangents and the curve; shew that the distance of its centre from P is

$$\frac{ab}{(a+b)\sec\frac{\omega}{2}+2\sqrt{(ab)}\tan\frac{\omega}{2}}$$

27. If two chords at right angles be drawn through a fixed point to meet a curve of the second degree, shew that

$$\frac{1}{Rr}+\frac{1}{R'r'}$$

is constant, where R and r are the segments of one chord made by the fixed point, and R' and r' those of the other.

28. The equation to the locus of the foci of all parabolas whose chords of contact with axes inclined at an angle α cut off a triangle of constant area is

$$r = k\sqrt{\{\sin\theta\sin(\alpha-\theta)\}}.$$

29. A parabola slides between two rectangular axes, find the curve traced out by the focus.

30. A parabola slides between two rectangular axes, find the curve traced out by the vertex.

31. Successive circles are drawn each touching the preceding one externally and each having double contact with a given parabola; shew that their radii form an arithmetical progression whose common difference is the latus rectum.

32. A system of ellipses is represented by the equation in rectangular co-ordinates

$$ax^2+2cxy+by^2=n(a+b),$$

where a, b, c are variable and n constant; shew that every parallelogram constructed on a pair of perpendicular diameters as diagonals will circumscribe a certain fixed circle.

33. If from any point in the tangent to a conic section a perpendicular be dropped upon the line joining the focus and the point of contact, prove that the distance of the point in the tangent from the directrix is to the distance of the foot of the perpendicular from the focus as $1 : e$.

34. Upon a given straight line as latus rectum, let any number of conic sections be drawn, and from the focus let two straight lines be drawn intersecting them all; then the chords of all the intercepted arcs will, if produced, pass through a single point.

35. A line of constant length moves so that its ends always lie on two given lines; find the locus traced out by a point in the line which divides it in a given ratio.

36. In any conic section if r and r' be focal distances at right angles to each other, and l be half the latus rectum, then

$$\left(\frac{1}{r}-\frac{1}{l}\right)^2+\left(\frac{1}{r'}-\frac{1}{l}\right)^2 \text{ is constant.}$$

37. Two conic sections equal in every respect are placed with their axes at right angles and with a common focus S; SP, SQ being radii vectores of the one and the other at right angles to each other, find the locus of the intersection of the tangents at P and Q.

Also find the locus when SPQ is a straight line.

38. S and H are the foci of an ellipse, and round S, H, as focus and centre, another ellipse is described, having its minor axis equal to the latus rectum of the former. Through any point P in the first draw SPQ to meet the second; it is required to find the locus of the intersection of HP and the ordinate QM.

39. A and B are the centres of two equal circles; AP, BQ, radii of these circles at right angles. If $AB^2 = 2AP^2$, the line PQ always passes through one of the points of intersection of the circles.

40. Tangents are drawn to a conic section at the points P, R; another tangent is drawn at an intermediate point Q, and meets the other tangents in M, N; shew that the angle MSN is half the angle PSR, S being a focus.

41. In a parabola the angle between any two tangents is half the angle subtended at the focus by the chord of contact.

42. A triangle is formed by the intersections of three tangents to a parabola; shew that the circle which circumscribes this triangle passes through the focus.

43. Given a focus and two tangents to a conic section, shew that the chord of contact passes through a fixed point.

44. A circle is described upon the minor axis of an ellipse as diameter; find the locus of the pole with respect to the ellipse of a tangent to the circle.

45. In a parabola two focal chords PSp, QSq, are drawn; shew that a focal chord parallel to PQ will meet pq produced on the tangent at the vertex.

46. If from the vertex of a parabola a pair of chords be drawn at right angles to each other, and on them a rectangle be completed, prove that the locus of the further angle is another parabola.

47. From a point P in the circumference of an ellipse chords PQ, PR are drawn at right angles; express the co-ordinates of the point of intersection of QR with the normal at P in terms of the co-ordinates of P. Shew that as P moves along the ellipse this point of intersection will describe the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2.$$

48. Shew that the locus of the centre of an equilateral hyperbola described about a given equilateral triangle is the circle inscribed in the triangle.

49. Two equal parabolas have the same axis and vertex, but are turned in opposite directions; chords of one parabola are tangents to the other; shew that the locus of the middle points of the chords is a parabola whose latus rectum is one-third of that of the given parabola.

50. The co-ordinates of the focus of the parabola whose equation when referred to two tangents inclined at an angle ω is $\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left(\frac{y}{b}\right)} = 1$, are

$$\frac{ab^2}{a^2 + b^2 + 2ab\cos\omega}, \text{ and } \frac{a^2b}{a^2 + b^2 + 2ab\cos\omega}.$$

51. If $ax^2 + 2bxy + cy^2 + 2a'x + 2c'y + d = 0$ be the equation to a parabola, the axis of the parabola will be given by the equation

$$(a+b)\left(x + \frac{a'}{a+c}\right) + (b+c)\left(y + \frac{c'}{a+c}\right) = 0.$$

52. Two equal parabolas have the same focus and their axes are at right angles to each other, and a normal to one of them is perpendicular to a normal to the other; prove that the locus of the intersection of such normals is a parabola.

53. Find the locus of the intersection of two normals in an ellipse which are at right angles.

54. Normals are drawn at the extremities of the conjugate diameters of an ellipse, and by their intersections form a parallelogram. If ϕ denote the excentric angle of an extremity of one of the conjugate diameters, shew that the area of the parallelogram is

$$\frac{8(a^2 - b^2)^2}{ab} \sin^3\phi \cos^3\phi.$$

55. Through the four angular points of a given square a circle is drawn, and also a series of curves of the second order, and common tangents to the circle and each curve are drawn. Find the locus of the points of contact of each curve with its tangent.

56. From any point T outside an ellipse two tangents TP and TQ are drawn to the ellipse; shew that a circle can be described with T as centre so as to touch SP, HP, SQ, HQ, or these lines produced.

If x and y are the co-ordinates of T, shew that the radius of the circle is

$$\frac{\sqrt{(a^2y^2 + b^2x^2 - a^2b^2)}}{a}.$$

CHAPTER XV.

ABRIDGED NOTATION.

301. *Through five points, no three of which are in one straight line, one conic section and only one can be drawn.*

Let the axis of x pass through two of the five points, and the axis of y through two of the remaining three points. Let the distances of the first two points from the origin be h_1, h_2, respectively, and those of the second two points k_1, k_2, respectively; also let h, k be the co-ordinates of the remaining point. Suppose (Art. 269)

$$ax^2 + bxy + cy^2 + dx + ey + 1 = 0 \quad\text{..........}(1)$$

to be the equation to a conic section passing through the five points. Since the curve passes through the points $(h_1, 0)$ $(h_2, 0)$, we have from (1)

$$ah_1^2 + dh_1 + 1 = 0 \quad\text{..........}(2),$$

$$ah_2^2 + dh_2 + 1 = 0 \quad\text{..........}(3).$$

Similarly, since the curve passes through $(0, k_1)$, $(0, k_2)$, we have

$$ck_1^2 + ek_1 + 1 = 0 \quad\text{..........}(4),$$

$$ck_2^2 + ek_2 + 1 = 0 \quad\text{..........}(5).$$

Lastly, since the curve passes through (h, k), we have

$$ah^2 + bhk + ck^2 + dh + ek + 1 = 0 \quad\text{..........}(6).$$

From (2) and (3) we find

$$a = \frac{1}{h_1 h_2}, \quad d = -\frac{h_1 + h_2}{h_1 h_2}.$$

From (4) and (5) we find

$$c = \frac{1}{k_1 k_2}, \quad e = -\frac{k_1 + k_2}{k_1 k_2};$$

then from (6) we can determine the value of b. Since no three of the five given points are in the same straight line, none of the quantities h_1, h_2, k_1, k_2, h, k, can be zero; hence the values of the coefficients a, b, c, d, e are all finite. If we substitute these values in (1), we obtain the equation to a conic section passing through the five given points. As each of the quantities a, b, c, d, e, has only *one* value, only *one* conic section can be made to pass through the five given points.

302. The investigation of the preceding article may still be applied when *three* of the given points are in one straight line; the point (h, k) for instance may be supposed to lie on the line joining $(0, k_1)$ and $(h_1, 0)$; the conic section in this case cannot be an ellipse, parabola, or hyperbola, since these curves cannot be cut by a straight line in more than two points; the conic section must therefore reduce to two straight lines, namely the line joining the three points already specified, and the line joining the other two points. If, however, *four* of the given points are in one straight line, the method of the preceding article is inapplicable; it is obvious that more than one pair of straight lines can then be made to pass through the five points.

303. We shall now give some useful forms of the equations to conic sections passing through the angular points of a triangle or touching its sides.

Let $u = 0$, $v = 0$, $w = 0$ be the equations to three straight lines which meet and form a triangle; the equation

$$lvw + mwu + nuv = 0 \text{.......................} (1),$$

where l, m, n are constants, will represent a conic section described round the triangle; also by giving suitable values to l, m, n, the above equation may be made to represent *any* conic section described round the triangle. This we proceed to prove.

I. The equation (1) is of the *second degree* in the variables x and y, which occur in the expressions u, v, w; hence (1) must represent a conic section.

II. The equation (1) is satisfied by the values of x and y, which make simultaneously $v=0$, $w=0$; the conic section therefore passes through the intersection of the lines represented by $v=0$ and $w=0$. Similarly the conic section passes through the intersection of $w=0$ and $u=0$, and also through the intersection of $u=0$ and $v=0$. Hence the conic section represented by (1) is described round the triangle formed by the intersection of the lines represented by $u=0$, $v=0$, $w=0$.

III. By giving suitable values to l, m, n, the equation (1) will represent any conic section described round the triangle. For let S denote a given conic section described round the triangle; take two points on S; suppose h_1, k_1 the co-ordinates of one of these points, and h_2, k_2 those of the other. If we first substitute h_1 and k_1 for x and y respectively in (1), and then substitute h_2 and k_2, we have two equations from which we can find the values of $\frac{m}{l}$ and $\frac{n}{l}$; suppose $\frac{m}{l}=p$ and $\frac{n}{l}=q$. Substitute these values in (1), which becomes

$$vw+pwu+quv=0\text{..........}(2);$$

this is therefore the equation to a conic section which has *five points* in common with S, namely, the three angular points of the triangle and the points (h_1, k_1), (h_2, k_2). The conic section (2) must therefore coincide with S by Art. 301. Hence the assertion is proved.

We might replace one of the constants in (1) by unity, but we retain the more symmetrical form; (1) may be written

$$\frac{l}{u}+\frac{m}{v}+\frac{n}{w}=0.$$

304. Equation (1) of the preceding article may be written

$$w(lv+mu)+nuv=0\text{..........}(1);$$

we will now determine where (1) meets the straight line represented by

$$lv + mu = 0 \quad\text{.........................}(2).$$

By combining (2) with (1) we deduce $nuv = 0$; therefore either $u = 0$, or $v = 0$; but by taking either of these suppositions and making use of (2), we see that the other supposition must also hold; hence the line (2) meets the curve (1) in only *one* point, namely, the point of intersection of $u = 0$ and $v = 0$.

Hence (2) is the *tangent* to (1) at this point. Similarly $mw + nv = 0$ is the tangent to (1) at the point of intersection of $w = 0$ and $v = 0$, and $nu + lw = 0$ is the tangent at the point of intersection of $u = 0$ and $w = 0$.

305. The demonstration of the preceding article is imperfect, because we know from Arts. 132, 222, that a line parallel to the axis of a parabola or to either asymptote of an hyperbola meets the curve in only one point, but is not the tangent at that point. The proposition may however be established in the following manner. Take the axis of x coincident with the line $u = 0$, so that u becomes qy, where q is some constant; also take the axis of y coincident with the line $v = 0$, so that v becomes px, where p is some constant. Suppose $w = Ax + By + C$. Then (1) of the preceding article becomes

$$(Ax + By + C)(lpx + mqy) + npqxy = 0.$$

By Art. 283 the equation to the tangent at the origin, that is, at the intersection of $x = 0$ and $y = 0$, is $lpx + mqy = 0$, or $lv + mu = 0$; which was to be proved.

306. Let each of the three tangents in Art. 304 be produced to meet the opposite side of the triangle formed by the lines $u = 0$, $v = 0$, $w = 0$; then it may be shewn that the three points of intersection lie on the straight line

$$\frac{u}{l} + \frac{v}{m} + \frac{w}{n} = 0.$$

The lines joining the angular points of the triangle formed by the tangents with the angular points of the original

triangle respectively opposite to them, are represented by the equations

$$\frac{u}{l}-\frac{v}{m}=0, \quad \frac{v}{m}-\frac{w}{n}=0, \quad \frac{w}{n}-\frac{u}{l}=0;$$

these three lines meet in a point. Thus when a triangle is inscribed in a conic section the lines joining each point with the pole of the opposite side meet in a point.

307. Let $u=0$, $v=0$, $w=0$ be the equations to three straight lines, then the equation

$$Au^2+Bv^2+Cw^2+2A'vw+2B'wu+2C'uv=0$$

will generally represent any assigned conic section, if the constants A, B, C, A', B', C' are properly determined.

For suppose we divide the equation by one of the constants as C', there are then five independent constants left. Now let S denote any assigned conic section; take five points in S and substitute the co-ordinates of the five points successively in the above equation; we shall thus have five equations for determining the five constants. Suppose a, b, c, a', b' the values thus determined, then the equation

$$au^2+bv^2+cw^2+2a'vw+2b'wu+2uv=0$$

represents a conic section which has five points in common with S, and which therefore coincides with S. (Art. 301.)

308. The method of the preceding article, although important and instructive, is not satisfactory, because we have not proved that the five equations from which the constants are to be determined are *consistent* and *independent*. There may be exceptions to the theorem, and we therefore use the word *generally* in the enunciation. If the three straight lines *meet in a point*, then the curve denoted by the equation always passes through that point, and the equation in this case will *not* represent *any assigned conic section*. If the three straight lines are parallel, u, v, w take the forms

$$lx+my+p, \qquad lx+my+p', \qquad lx+my+p'',$$

and the equation takes the form

$$\lambda\,(lx+my)^2+\mu\,(lx+my)+\nu=0,$$

which represents two parallel straight lines, and thus will *not* represent *any assigned conic section*. With these exceptions, however, the theorem is universally true, as we shall now shew by another demonstration.

Since the lines are not all parallel, two of them at least will meet; suppose $u=0$ and $v=0$ to be these two, and take their directions for the axes of y and x respectively; then $u=0$ becomes $x=0$, and $v=0$ becomes $y=0$; also $w=0$ may be written $lx+my+n=0$. We have then to shew that the equation

$$Ax^2+By^2+C(lx+my+n)^2+2A'y(lx+my+n) \\ +2B'x(lx+my+n)+2C'xy=0 \ldots\ldots (1)$$

will represent any assigned conic section by properly determining the constants A, B, &c. Suppose

$$ax^2+2bxy+cy^2+2dx+2ey+f=0 \ldots\ldots\ldots (2)$$

to be the equation to the assigned conic section. Arrange the terms in (1) and equate the coefficients of the corresponding terms in (1) and (2); thus

$$Cn^2=f, \quad A'n+Cmn=e, \quad B'n+Cln=d,$$
$$B+Cm^2+2A'm=c, \quad A+Cl^2+2B'l=a,$$
$$Clm+A'l+B'm+C'=b.$$

These equations determine successively C, A', B', B, A, C'. As the given lines do not meet in a point, n is not zero; hence the values found for C, A', &c. are all finite and determinate. Thus (1) is shewn to coincide with (2), and the required theorem is proved.

309. *To express the equation to a conic section which touches the sides of a triangle.*

Let $u=0$, $v=0$, $w=0$ be the equations to the sides of a triangle; then any conic section may be represented by the equation

$$Au^2+Bv^2+Cw^2+2A'vw+2B'wu+2C'uv=0 \ldots\ldots (1).$$

To find where this conic section meets the line $u=0$, we must put $u=0$; thus (1) becomes

$$Bv^2+Cw^2+2A'vw=0 \ldots\ldots\ldots\ldots\ldots (2).$$

Now from (2) we obtain by solution *two* values of $\frac{v}{w}$, say $\frac{v}{w}=\mu_1$, and $\frac{v}{w}=\mu_2$. The equation $v=\mu_1 w$ represents some straight line passing through the intersection of $v=0$, and $w=0$. Hence since (1) is satisfied by those values of x and y which make simultaneously $u=0$ and $v-\mu_1 w=0$, the intersection of the lines $u=0$ and $v-\mu_1 w=0$ is a point on (1). Similarly the intersection of $u=0$ and $v-\mu_2 w=0$ is a point on (1). Hence the line $u=0$ will meet (1) in *two* points, and therefore will not be a tangent to it, unless the lines

$$v-\mu_1 w=0, \text{ and } v-\mu_2 w=0,$$

coincide. Hence that $u=0$ may *touch* (1) we must have $\mu_1=\mu_2$, and therefore $A'^2=BC$.

Similarly that $v=0$ may touch (1) we must have $B'^2=CA$; and that $w=0$ may touch (1) we must have $C'^2=AB$. From these three relations we see that A, B, and C must have the *same* sign, because the product of each two is positive. Also the sign of A, B, and C may be supposed positive, because if each of them were negative we could change the sign of every term in (1), and thus make the coefficients of u^2, v^2, and w^2 positive. We may therefore put

$$A=l^2, \quad B=m^2, \quad C=n^2;$$

thus

$$A'=\pm mn, \quad B'=\pm nl, \quad C'=\pm lm.$$

Hence (1) becomes

$$l^2u^2+m^2v^2+n^2w^2\pm 2mnvw \pm 2nlwu \pm 2lmuv=0 \ldots\ldots (3).$$

We shall now examine the ambiguity of signs that appears in this expression.

I. Suppose all the upper signs to be taken. The equation may then be written

$$(lu+mv+nw)^2=0.$$

This is the equation to a straight line, or rather to two coincident straight lines.

II. Suppose the lower sign to be taken twice and the upper sign once; we have then three cases,

$$(lu + mv - nw)^2 = 0, \quad \text{or} \quad (lu - mv + nw)^2 = 0,$$
$$\text{or} \quad (-lu + mv + nw)^2 = 0.$$

Each equation represents two coincident straight lines.

III. Since then the forms in I. and II. represent straight lines, we see by excluding these cases from (3), that if *a curve of the second degree touch the straight lines*

$$u = 0, \quad v = 0, \quad w = 0,$$

its equation must take one of the forms

$$l^2u^2 + m^2v^2 + n^2w^2 - 2mnvw - 2nlwu - 2lmuv = 0 \ldots (4),$$
$$l^2u^2 + m^2v^2 + n^2w^2 - 2mnvw + 2nlwu + 2lmuv = 0 \ldots (5),$$
$$l^2u^2 + m^2v^2 + n^2w^2 + 2mnvw - 2nlwu + 2lmuv = 0 \ldots (6),$$
$$l^2u^2 + m^2v^2 + n^2w^2 + 2mnvw + 2nlwu - 2lmuv = 0 \ldots (7).$$

These four forms may also be written

$$\sqrt{(lu)} + \sqrt{(mv)} + \sqrt{(nw)} = 0 \ldots\ldots (8) \text{ from } (4),$$
$$\sqrt{(-lu)} + \sqrt{(mv)} + \sqrt{(nw)} = 0 \ldots\ldots (9) \ \ldots \ (5),$$
$$\sqrt{(lu)} + \sqrt{(-mv)} + \sqrt{(nw)} = 0 \ldots\ldots (10) \ \ldots \ (6),$$
$$\sqrt{(lu)} + \sqrt{(mv)} + \sqrt{(-nw)} = 0 \ldots\ldots (11) \ \ldots \ (7),$$

which may be verified by transposing and squaring, so as to put the equations in a rational form.

310. It is easy to verify the proposition that the curve represented by the equation

$$\sqrt{(lu)} + \sqrt{(mv)} + \sqrt{(nw)} = 0$$

cannot cut the lines $u = 0$, $v = 0$, $w = 0$. For suppose the

above equation satisfied by the co-ordinates of a point; then these co-ordinates must make lu, mv, and nw, *all positive, or all negative.* Suppose lu is positive; then for any point on the other side of $u=0$, the expression lu becomes negative, and thus the co-ordinates of such a point will *not* satisfy the equation unless both mv and nw are also negative. But if the curve cuts the line $u=0$, there will be points on both sides of $u=0$ lying on the curve, and it will be possible to change the sign of u without changing the signs of v and w. Hence the curve cannot cut the line $u=0$. Similarly it cannot cut the lines $v=0$, $w=0$.

The same mode of proof will shew that the curves represented by equations (9), (10), and (11), of the preceding article cannot cut the lines $u=0$, $v=0$, $w=0$.

311. The forms in equations (5), (6), and (7) of Art. 309 may be derived from (4) by changing the sign of one of the constants. Thus, for example, (5) may be derived from (4) by changing the sign of l. In the following article we shall use (4) as the equation to a conic section touching the sides of a triangle; it will be found that we might have used (5), (6), or (7). We shall see in a subsequent article, a case in which it is necessary to distinguish the forms. See Arts. 314, 315.

312. Equation (4) of Art. 309 may be written

$$(lu-mv)^2+nw\,(nw-2mv-2lu)=0 \;\ldots\ldots\; (1).$$

If we combine this with $w=0$, we deduce that

$$lu-mv=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2);$$

hence we can interpret the last equation; it represents a line passing through the intersection of $u=0$ and $v=0$, and also through the point where the line $w=0$ meets the curve (1). It may be shewn as in Art. 304, that

$$nw-2mv-2lu=0 \;\ldots\ldots\ldots\ldots\; (3)$$

represents the tangent to (1) at the other point where (2) meets it.

Similarly we can interpret

$$mv - nw = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

$$lu - 2nw - 2mv = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$

$$nw - lu = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

$$mv - 2lu - 2nw = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

The intersection of (3) with $w = 0$, of (5) with $u = 0$, and of (7) with $v = 0$ will lie on the line

$$lu + mv + nw = 0.$$

The line $lu + mv = 0$ passes through the intersection of $u = 0$ and $v = 0$, and also through the intersection of (3) and $w = 0$; hence its position is known.

Similarly $mv + nw = 0$, and $nw + lu = 0$, can be interpreted.

313. *To find the equation to the circle described round a triangle.*

It will be convenient in this and the two following articles to use the form

$$x \cos \alpha + y \sin \alpha - p = 0$$

as the type of the equation to a straight line; we shall therefore put α, β, γ for u, v, w respectively (Art. 73).

Let $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be the equations to the sides of a triangle; then, by Art. 303,

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

will represent *any* conic section described round the triangle; hence by giving proper values to l, m, n, this equation may be made to represent the circle which we know by geometry can be described round the triangle. We might proceed thus: in (1) write for α, β, γ the expressions which they represent, then equate the coefficient of xy to zero, and the coefficient of x^2 to that of y^2; we shall thus have two equations for determining $\frac{n}{l}$ and $\frac{m}{l}$; and with the values thus

obtained (1) will represent the required circle. We leave this as an exercise for the student, and adopt another method. The equation to the tangent to (1) at the intersection of $\alpha=0$, and $\beta=0$, is, by Art. 304,

$$l\beta+m\alpha=0 \;\dots\dots\dots\dots\dots\dots\dots\dots\; (2).$$

Let A, B, C denote the angles of the triangle opposite the sides $\alpha=0$, $\beta=0$, $\gamma=0$, respectively; by Euclid, III. 32, the tangent denoted by (2) must make an angle A with the line $\alpha=0$, and an angle B with the line $\beta=0$. Suppose the origin of co-ordinates *within* the triangle, then the equation to the line passing through the intersection of $\alpha=0$ and $\beta=0$, and making angles A and B respectively with these lines, is

$$\alpha \sin B+\beta \sin A=0 \;\dots\dots\dots\dots\dots\dots\; (3).$$

Thus (2) must coincide with (3); therefore

$$\frac{l}{m}=\frac{\sin A}{\sin B}.$$

Similarly, $$\frac{m}{n}=\frac{\sin B}{\sin C}.$$

Thus the equation to the circle described round the triangle is

$$\beta\gamma \sin A+\gamma\alpha \sin B+\alpha\beta \sin C=0.$$

314. *To find the equation to the circle inscribed in a triangle.*

Suppose the origin of co-ordinates *within* the triangle; then for all points on the circle α, β, γ are *negative* quantities (see Art. 54). Now the equation to the circle must be of one of the forms (8), (9), (10), (11) given in Art. 309; the first is the only form applicable, namely,

$$\sqrt{(l\alpha)}+\sqrt{(m\beta)}+\sqrt{(n\gamma)}=0 \;\dots\dots\dots\dots\dots\; (1),$$

which is equivalent to

$$\sqrt{(-l\alpha)}+\sqrt{(-m\beta)}+\sqrt{(-n\gamma)}=0 \;\dots\dots\dots\dots\; (2).$$

The other forms are inapplicable, because they would introduce impossible expressions. We have then to determine the values of l, m, and n. If we put $\alpha = 0$ in (1), we obtain $\frac{\beta}{\gamma} = \frac{n}{m}$; thus $\frac{n}{m}$ is the ratio of the perpendiculars drawn to the sides $\beta = 0$, $\gamma = 0$, respectively, from the point where the circle meets the line $\alpha = 0$. Let r be the radius of the circle; then we know from geometry that the perpendicular from this point on $\beta = 0$ is

$$r \cot \frac{C}{2} \sin C \text{ or } 2r \cos^2 \frac{C}{2};$$

a similar expression holds for the perpendicular on $\gamma = 0$. Hence

$$\frac{n}{m} = \frac{\cos^2 \frac{C}{2}}{\cos^2 \frac{B}{2}}.$$

Similarly $$\frac{l}{n} = \frac{\cos^2 \frac{A}{2}}{\cos^2 \frac{C}{2}}.$$

Hence the required equation is

$$\cos \frac{A}{2} \sqrt{\alpha} + \cos \frac{B}{2} \sqrt{\beta} + \cos \frac{C}{2} \sqrt{\gamma} = 0.$$

315. *To find the equation to the circle which touches one side of a triangle and the other two sides produced.*

Let the circle be required to touch the side opposite to the angle A and the other two sides produced. Suppose the origin *within* the triangle; then for all points comprised between the side $\alpha = 0$ and the other sides produced, α is positive and β and γ are negative. Hence by Art. 309, the form of the equation to the circle must be

$$\sqrt{(-l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0.$$

Hence, as before, by considering the point where the circle meets the line $\alpha = 0$, we have

$$\frac{n}{m} = \frac{\cos^2 \frac{\pi - C}{2}}{\cos^2 \frac{\pi - B}{2}} = \frac{\sin^2 \frac{C}{2}}{\sin^2 \frac{B}{2}},$$

and
$$\frac{l}{n} = \frac{\cos^2 \frac{A}{2}}{\cos^2 \frac{\pi - C}{2}} = \frac{\cos^2 \frac{A}{2}}{\sin^2 \frac{C}{2}}.$$

Hence the required equation is

$$\cos \frac{A}{2} \sqrt{(-\alpha)} + \sin \frac{B}{2} \sqrt{\beta} + \sin \frac{C}{2} \sqrt{\gamma} = 0.$$

Similarly the equations to the other two circles may be written down.

316. The results in Arts. 306 and 312 which hold for any conic section, will of course hold for a circle inscribed in, or described about, a triangle respectively. We have only to use the values of l, m, n, found in Arts. 313—315.

317. Let there be any quadrilateral, and let its sides be represented by the equations

$$t = 0, \quad u = 0, \quad v = 0, \quad w = 0,$$

then the equation
$$tu + kvw = 0,$$

where k is a constant, represents a conic section circumscribing the quadrilateral. For the equation represents a conic section passing through the four points determined respectively by

$$t = 0 \text{ and } v = 0, \qquad t = 0 \text{ and } w = 0,$$
$$u = 0 \text{ and } v = 0, \qquad u = 0 \text{ and } w = 0.$$

Also by giving a suitable value to k, the equation may be made to represent *any* conic section passing through these four points.

The above equation has the following geometrical interpretation. If any quadrilateral figure be inscribed in a conic section, the product of the perpendiculars drawn from any point of the curve on two opposite sides bears a constant ratio to the product of the perpendiculars on the other two sides.

We may observe that the term *quadrilateral* is often used in analytical geometry in a wider sense than in ordinary synthetical geometry. Thus, if we have four given points, we may obtain three different quadrilaterals by connecting these points in different ways. Take, for example, the figure in Art. 76; and let A, B, C, D be the given points. The three different quadrilaterals are (1) the figure bounded by AB, BC, CD, DA; (2) the figure bounded by AC, CD, DB, BA; which in fact consists of the two triangles GAB and GCD; (3) the figure bounded by AC, CB, BD, DA, which in fact consists of the two triangles GBC and GDA.

Similarly, four given straight lines may be considered to form three different quadrilaterals by their intersections. Take, for example, the figure in Art. 76, and let the given lines be EDC, EAB, AGC, BGD. The three different quadrilaterals are (1) the figure bounded by GC, CE, EB, BG; (2) the figure bounded by GD, DE, EA, AG; (3) the figure bounded by AC, CD, DB, BA.

If four lines have for their equations

$$t=0, \quad u=0, \quad v=0, \quad w=0,$$

the conic sections passing through the angular points of the three different quadrilaterals which these lines form, may be denoted by the equations

$$tu+k_1 vw=0, \quad tv+k_2 uw=0, \quad tw+k_3 uv=0.$$

318. We shall next consider the equation

$$uv+kw^2=0.$$

This represents a conic section which passes through the point determined by $u=0$ and $w=0$, and also through the point determined by $v=0$ and $w=0$. Also each of the lines $u=0$ and $v=0$ *touches* the conic section where it meets it; for if we combine $u=0$ with the above equation, we see that $w=0$

also, that is, the line $u=0$, meets the curve in only *one* point, namely, that point in which $u=0$ and $w=0$ intersect. Similarly the line $v=0$ touches the curve. Thus $u=0$ and $v=0$ represent two tangents to the conic section, and $w=0$ represents the corresponding chord of contact.

We may also shew in the following way that the line $u=0$ cannot *cut* the curve; for points on one side of the line $u=0$, the expression u is positive, and for points on the other side of the line, negative; but kw^2 is of invariable sign; thus $u=0$ cannot cut the curve.

The geometrical interpretation of the above equation is as follows. The product of the perpendiculars from any point of a conic section on a pair of tangents bears a constant ratio to the square of the perpendicular from the same point on the chord of contact.

319. Next take the equation

$$l^2u^2+m^2v^2=n^2w^2.$$

This may be written

$$(nw+mv)(nw-mv)=l^2u^2.$$

Hence by the preceding article

$$nw+mv=0 \text{ and } nw-mv=0$$

are tangents to the conic section represented by the equation, and $u=0$ is the equation to the corresponding chord of contact. Since these two tangents meet in the point of intersection of $v=0$ and $w=0$, it follows that this point is the *pole* of $u=0$.

Similarly we may write the equation in the form

$$(nw+lu)(nw-lu)=m^2v^2,$$

and infer that the point of intersection of $u=0$ and $w=0$ is the pole of $v=0$.

Hence it follows that the point of intersection of $u=0$ and $v=0$ is the pole of $w=0$. See Art. 291.

320. The following is a particular case of the preceding article,

$$\alpha^2+\beta^2=n^2\gamma^2. \quad \text{(See Art. 73.)}$$

Suppose the lines $\alpha=0$, $\beta=0$, at right angles; then $\alpha^2+\beta^2$ is the square of the distance of the point (x, y) from the intersection of $\alpha=0$ and $\beta=0$. Hence the above equation represents a conic section which has $\gamma=0$ for its directrix, and the intersection of $\alpha=0$ and $\beta=0$ for its focus. The lines

$$n\gamma-\alpha=0 \text{ and } n\gamma+\alpha=0$$

are tangents to the conic section touching it at the extremities of the focal chord $\beta=0$; also these tangents meet in the line $\gamma=0$; hence, *the tangents at the extremities of any focal chord meet in the corresponding directrix.* Also the above tangents meet on the line $\alpha=0$, which by supposition is perpendicular to $\beta=0$; hence, *the line which joins the focus to the intersection of tangents at the extremities of a focal chord is perpendicular to that focal chord.*

321. If $u=0$ and $v=0$ be the equations to two conic sections which meet in four points, then $u+lv=0$ will represent any conic section which passes through the four points of intersection. This will be obvious after the proofs given of similar propositions.

Also if $w=0$ and $w'=0$ be the equations to two straight lines, $u+lww'=0$ will represent any conic section passing through the four points in which the lines $w=0$ and $w'=0$ meet the conic section $u=0$.

Also $u+lw^2=0$ will represent a conic section passing through the points of intersection of the conic section $u=0$, and the line $w=0$. This conic section will have the same tangent as $u=0$ at the points where $u=0$ and $w=0$ intersect; we might anticipate this would be the case from observing the interpretation of the equation $u+lww'=0$, and supposing the line $w'=0$ to approach the line $w=0$, and ultimately to coincide with it. We may prove it strictly by taking one of the points where $u=0$ meets $w=0$ for the origin, and the line $w=0$ for the axis of x; thus u becomes of the form

$$Ax^2+Bxy+Cy^2+Dx+Ey,$$

and we can see, by Art. 283, that

$$Ax^2+Bxy+Cy^2+Dx+Ey=0$$

and $$Ax^2 + Bxy + Cy^2 + Dx + Ey + ly^2 = 0$$

have the same tangent at the origin.

Also by giving a suitable value to l the equation $u + lw^2 = 0$ may be made to represent the two straight lines which touch the conic section $u = 0$ at the points where it intersects the straight line $w = 0$. This may be inferred from Art. 293; the equation $w = 0$ is equivalent to the equation

$$\frac{x}{h} + \frac{y}{k} - 1 = 0,$$

and the equation $u = 0$ is equivalent to

$$\left(\frac{x}{h} + \frac{y}{k} - 1\right)^2 + \mu xy = 0.$$

Thus by taking $l = -1$ we have $u + lw^2 = \mu xy$; and the equation $xy = 0$ denotes the two tangents to the conic section $u = 0$ at its points of intersection with the straight line $w = 0$.

322. Pascal's Theorem. *The three intersections of the opposite sides of any hexagon inscribed in a conic section are in one straight line.*

Let $r = 0$, $s = 0$, $t = 0$, $u = 0$, $v = 0$, $w = 0$,

be the equations to the sides of a hexagon which is inscribed in the conic section $S = 0$. Let the hexagon be divided by a new line $\phi = 0$ into two quadrilaterals, one of which has for its sides the lines obtained by equating to zero successively, r, s, t, ϕ, and the other the lines obtained by equating to zero successively, u, v, w, ϕ. Now we know that if a, b, l, m are appropriate constants, the equation to the conic section may be written in the forms

$$as\phi + brt = 0 \text{ and } lv\phi + muw = 0;$$

therefore $as\phi + brt$ and $lv\phi + muw$ must each be identical with S; therefore

$$as\phi + brt = lv\phi + muw;$$

$$\therefore (as - lv)\,\phi = muw - brt.$$

The right-hand member of this equation vanishes when u and r simultaneously vanish, and when u and t simultaneously vanish; also when w and r simultaneously vanish, and when w and t simultaneously vanish. Since the left-hand member is identically equal to the right-hand, the left-hand member must also vanish in these four cases; that is, one of its two factors ϕ and $as - lv$ must vanish in each of these four cases. By construction, $\phi = 0$ represents the line joining the point determined by $r = 0$ and $w = 0$, with the point determined by $t = 0$ and $u = 0$; and thus we see that $as - lv = 0$ is the line joining the intersection of $u = 0$ and $r = 0$ with that of $t = 0$ and $w = 0$. But the line $as - lv = 0$ obviously passes through the intersection of $s = 0$ and $v = 0$; therefore the three points determined respectively by

$$u = 0 \text{ and } r = 0, \quad t = 0 \text{ and } w = 0, \quad s = 0 \text{ and } v = 0,$$

lie on a straight line.

It is to be observed that if six points be connected by straight lines in different ways, as many as sixty figures can be formed which may be called *hexagons* in an extended sense of that word. Thus for six given points on a conic section there will be sixty applications of Pascal's Theorem.

323. Let $s = 0$ be the equation to a conic section, and

$$u = 0, \quad v = 0, \quad w = 0,$$

equations to three straight lines; then

$$s - l^2u^2 = 0, \quad s - m^2v^2 = 0, \quad s - n^2w^2 = 0,$$

represent curves of the second degree touching the proposed conic section. By properly choosing u, v, w, l, m, n, we may make each of the last three equations represent a pair of straight lines touching $s = 0$. (See Art. 321.) Thus, if there be a hexagon circumscribed round the conic section $s = 0$, the equations

$$s - l^2u^2 = 0 \ldots (1), \quad s - m^2v^2 = 0 \ldots (2), \quad s - n^2w^2 = 0 \ldots (3),$$

may be taken to represent the six sides of the hexagon.

By combining (1) and (2) we obtain

$$s - l^2u^2 - (s - m^2v^2) = 0, \quad \text{or} \quad (mv - lu)(mv + lu) = 0 \ldots (4),$$

for the equation to a pair of lines which pass through the intersections of (1) and (2).

Similarly $$(nw - mv)(nw + mv) = 0 \ldots\ldots\ldots\ldots\ldots (5)$$

represents a pair of lines which pass through the intersections of (2) and (3). And

$$(lu - nw)(lu + nw) = 0 \ldots\ldots\ldots\ldots\ldots (6)$$

represents a pair of lines which pass through the intersections of (3) and (1).

The six lines which we have obtained may be arranged in four groups, each containing three lines which meet in a point, namely,

$$\begin{array}{lll} mv - lu = 0, & nw - mv = 0, & lu - nw = 0, \\ mv + lu = 0, & nw + mv = 0, & lu - nw = 0, \\ mv + lu = 0, & nw - mv = 0, & lu + nw = 0, \\ mv - lu = 0, & nw + mv = 0, & lu + nw = 0. \end{array}$$

This result is consistent with Brianchon's theorem; *if a hexagon be described about a conic section the three diagonals which join opposite angles meet in a point.*

For suppose that a hexagon is described round a conic section, and let its angular points be denoted by A, B, C, D, E, F. By properly choosing u, v, w, l, m, n, we may make equation (1) denote the lines AB and DE, equation (2) denote the lines BC and EF, and equation (3) denote the lines CD and FA. We will now examine what lines are determined by equations (4), (5), and (6). Equation (4) determines the two lines which pass through the intersections of the lines determined by (1) and (2); and as the signs of l and m are at present in our power we may take them so that $mv - lu = 0$ shall represent the line BE, and then $mv + lu = 0$ will represent the line joining the point which is common to AB and

EF with the point which is common to BC and DE. Similarly as the sign of n is still in our power, we may take it so that $nw - mv = 0$ shall represent the line CF, and then $nw + mv = 0$ will represent the line joining the point which is common to BC and FA with the point which is common to CD and EF. One of the two lines represented by (6) is AD, and the other is the line joining the point which is common to DE and FA with the point which is common to CD and AB; it is however not obvious how we are in general to discriminate between these two lines. Thus the proof of Brianchon's theorem is not perfectly satisfactory, and accordingly we shall give another proof by which the theorem is deduced from that of Pascal.

Let the angular points of the hexagon be denoted as before by the letters A, B, C, D, E, F. Let the line be drawn which passes through the points of contact of the conic section and the tangents AB, BC; also let the line be drawn which passes through the points of contact of the conic section and the tangents DE, EF; and let P denote the point which is common to these two lines. *Then P is the pole of BE*; see Arts. 103, 120, 289. In the same way we may determine the pole of CF which we shall denote by Q, and the pole of AD which we shall denote by R. By Pascal's theorem P, Q, and R lie in a straight line; hence CF, BE, and AD meet in a point, namely, in the pole of the line PQR; see Art. 291.

For further information on the subject of this chapter the student is referred to Salmon's *Conic Sections*.

EXAMPLES.

1. Shew that if $a - c : a' - c' :: b : b'$, a circle may be described through the intersections of the two conic sections

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

$$a'x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0.$$

Find also the condition that a parabola may be described passing through the origin and the points of intersection of these curves.

2. Two conic sections have their principal axes at right angles; shew that a circle will pass through their points of intersection.

3. The equations to two conic sections are

$$Ay^2 + 2Bxy + Cx^2 + 2A'x = 0,$$

$$ay^2 + 2bxy + cx^2 + 2a'x = 0.$$

Shew that the lines joining the origin with their points of intersection will be perpendicular to each other if

$$a'(A + C) = A'(a + c).$$

4. An ellipse is described so as to touch the asymptotes of an hyperbola; shew that two of the chords joining the points of intersection of the ellipse and hyperbola are parallel.

5. If $\alpha\beta = c^2$ be the equation to an hyperbola (Art. 73), then $\alpha\beta = 0$, $\alpha^2 - \beta^2 = 0$, $\alpha^2 - n^2\beta^2 = 0$, are the respective equations to the asymptotes, the axes, and a pair of conjugate diameters, n being any constant.

6. The straight lines which bisect the angles of a triangle, meet the opposite sides in the points P, Q, R, respectively; find the equation to an ellipse described so as to touch the sides of the triangle in these points.

7. From any point two straight lines are drawn, one inclined at an angle α, the other at an angle $\frac{\pi}{2} + \alpha$, to the axis of a parabola; shew that another parabola may be described which shall pass through the four points of intersection, whose axis is inclined at an angle 2α to that of the given parabola.

8. Prove that the equation to the conic section which passes through the point (h, k), and touches the parabola $y^2 = lx$ at the vertex and at an extremity of the latus rectum, is

$$(y^2 - lx)(k - 2h)^2 = (y - 2x)^2 (k^2 - lh).$$

Shew that it is an ellipse or hyperbola according as the point (h, k) is within or without the parabola.

9. A conic section touches the sides of a triangle ABC in the points a, b, c; and the straight lines Aa, Bb, Cc, intersect the conic in a', b', c'; shew that

(1) the lines Aa, Bb, Cc pass respectively through the intersections of Bc' and Cb', Ca' and Ac', Ab' and Ba',

(2) the intersections of the lines ab and $a'b'$, bc and $b'c'$, ac and $a'c'$, lie respectively in AB, BC, CA.

10. A conic section is described round a triangle ABC; lines bisecting the angles of this triangle meet the conic in the points A', B', C', respectively; express the equations to $A'B, A'C, A'B'$.

11. If a conic section be described about any triangle, and the points where the lines bisecting the angles of the triangle meet the conic be joined, the intersection of the sides of the triangle so formed with the corresponding sides of the original triangle lie in a straight line.

12. Interpret the equation

$$\left(\frac{x}{a}+\frac{y}{b}-1\right)\left(\frac{x}{a'}+\frac{y}{b'}-1\right)+\mu xy=0;$$

how many parabolas can be drawn through four given points?

13. If $u=0$, $v=0$, $w=0$ represent the sides of a triangle, shew that the sides of any triangle which has one angle on each side of the former may be represented by

$$u+nv+\frac{w}{m}=0,\quad \frac{u}{n}+v+lw=0,\quad mu+\frac{v}{l}+w=0,$$

where l, m, n are constants.

Find also the relation which must hold between l, m, n, in order that the lines joining corresponding angles of the two triangles may meet in a point.

14. A circle and a rectangular hyperbola intersect in four points, and one of their common chords is a diameter of the hyperbola; shew that another of them is a diameter of the circle.

15. ACA' is the major axis of an ellipse, P any point on the circle described on the major axis, AP, $A'P$ meet the ellipse in Q, Q'; shew that the equation to QQ' is

$$(a^2+b^2)\, y \sin\theta + 2b^2x\cos\theta - 2ab^2 = 0,$$

the ellipse being referred to its axes, and θ being the angle ACP.

If an ordinate to P meet QQ' in R, the locus of R is an ellipse.

16. The locus of a point such that the sum of the squares of the perpendiculars drawn from it to the sides of a given triangle shall be constant, is an ellipse; and if the constant be so chosen that the ellipse may touch the side opposite to the angle A in D, then

$$CD : BD :: b^2 : c^2.$$

17. With the notation of Art. 313, shew that the equation to the line through C and the centre of the circle is

$$\alpha \cos B = \beta \cos A.$$

18. Suppose in Art. 313 that D is the middle point of the arc AB; then the equations to BD and AD are respectively

$$\alpha \sin C + \gamma (\sin A + \sin B) = 0;$$

$$\beta \sin C + \gamma (\sin A + \sin B) = 0.$$

19. In Art. 309, equation (4), if A', B', C' be the points of contact of the triangle and conic section, shew that the equation to $A'B'$ is

$$lu + mv - nw = 0.$$

20. In the figure of Art. 292, suppose $u=0$ the equation to AC, $v=0$ the equation to BD, and $w=0$ the equation to EF, and that

$$l^2u^2 + m^2v^2 - n^2w^2 = 0$$

represents a conic section passing through A, B, C, D; then express the equations to the tangents at A, B, C, D, and also

to the lines AB, BC, CD, DA. Shew also that the line FG passes through the intersection of the tangents at A and B, and of those at C and D.

21. Find the condition that the line

$$\lambda u + \mu v + \nu w = 0$$

may *touch* the conic section

$$\sqrt{(lu)} + \sqrt{(mv)} + \sqrt{(nw)} = 0.$$

22. Give a geometrical interpretation of equation (1) in Art. 304, and shew that it is a particular case of the theorem in Art. 317.

23. Interpret the last equation in Art. 313; deduce the following theorem; if from any point of the circle which circumscribes a triangle, perpendiculars are drawn on the sides of the triangle, the feet of the perpendiculars lie in one straight line.

24. If ellipses be inscribed in a triangle each with one focus in a fixed straight line, the locus of the other focus is a conic section passing through the angular points of the triangle.

25. Three conic sections are drawn touching respectively each pair of the sides of a triangle at the angular points where they meet the third side, and each passing through the centre of the inscribed circle; shew that the three tangents at their common point meet the sides of the triangle which intersect their respective conics in three points lying in a straight line. Shew also that the common tangents to each pair of conics intersect the sides of the triangle which touch the several pairs of conics in the above three points.

26. With the angular points of a triangle ABC as centres, and the sides as asymptotes, three hyperbolas are described, having A', B', C' as their vertices respectively: prove that if

$$AA' \sin\frac{A}{2} = BB' \sin\frac{B}{2} = CC' \sin\frac{C}{2},$$

the intersections of each pair of hyperbolas lie on the axis of the third.

27. The necessary and sufficient condition in order that the equation $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$ may represent a rectangular hyperbola is $l + m + n = 0$.

28. Shew that $\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0$ represents in general an ellipse, parabola, or hyperbola according as

$$lmn\left(\frac{l}{a} + \frac{m}{b} + \frac{n}{c}\right)$$

is positive, zero, or negative; where a, b, c denote the lengths of the sides of the triangle formed by $\alpha = 0$, $\beta = 0$, $\gamma = 0$.

29. Shew that $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ represents in general an ellipse, parabola, or hyperbola according as

$$l^2a^2 + m^2b^2 + n^2c^2 - 2lmab - 2mnbc - 2nlca$$

is negative, zero, or positive.

30. Find the condition that the line

$$\lambda u + \mu v + \nu w = 0$$

may touch the conic section

$$lu^2 + mv^2 + nw^2 = 0.$$

31. Find the *fourth* point of intersection of the conic sections

$$lvw + mwu + nuv = 0,$$

and $$l'vw + m'wu + n'uv = 0.$$

32. Shew that the equation to the radical axis of the circles inscribed in a triangle and circumscribed about it is

$$\alpha \operatorname{cosec} A \cos^4\frac{A}{2} + \beta \operatorname{cosec} B \cos^4\frac{B}{2} + \gamma \operatorname{cosec} C \cos^4\frac{C}{2} = 0.$$

33. Find the equation to the diameter of the curve

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$$

which passes through the point of intersection of the lines $\beta = 0$ and $\gamma = 0$.

34. Find the equation to the tangent to the curve

$$\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0,$$

which is parallel to the line $\gamma = 0$; and thence shew that the centre of the curve is determined by

$$\frac{\alpha}{mc+nb} = \frac{\beta}{na+lc} = \frac{\gamma}{lb+ma}.$$

35. From a point P two tangents are drawn to a conic section meeting it in the points M and N respectively; the line through P which bisects the angle MPN meets the chord MN in Q; any chord of the conic section is drawn through Q; shew that the segments into which the chord is divided by the point Q subtend equal angles at P.

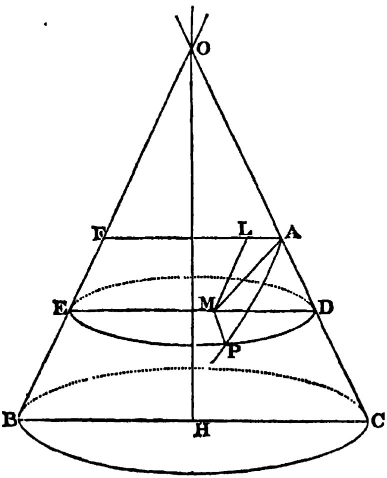
CHAPTER XVI.

SECTIONS OF A CONE. ANHARMONIC RATIO AND HARMONIC PENCIL.

Sections of a Cone.

324. We shall now shew that the curves which are included under the name *conic sections*, can be obtained by the intersection of a cone and a plane.

Def. A cone is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which remains fixed. The fixed side is called the axis of the cone.



Let OH be the fixed side, and OHC the right-angled triangle which revolves round OH. In order to obtain a

cone such as is considered in ordinary synthetical geometry, we should take only a *finite* line OC; but in analytical geometry it is usual to suppose OC *indefinitely produced both ways.* A section of the cone made by a plane through OH and OC will meet the cone in a line OB, which is the position OC would occupy after revolving half way round. Let a section of the cone be made by a plane perpendicular to the plane BOC; let AP be the section, A being the point where the cutting plane meets OC; we have to find the nature of this curve AP. Let a plane pass through any point P of the curve, and be perpendicular to the axis OH; this plane will obviously meet the cone in a circle DPE, having its diameter DE in the plane BOC. Let MP be the line in which the plane of this circle meets the plane section we are considering, M being in the line DE. Since each of the planes which intersect in MP is perpendicular to the plane BOC, MP is perpendicular to that plane, and therefore to every line in that plane.

Draw AF parallel to ED, and ML parallel to OB; join AM. Let $AM=x$, $MP=y$, $OA=c$, $HOC=\alpha$, $OAM=\theta$; the angle AML will be equal to the inclination of AM to OB, that is, to $\pi-\theta-2\alpha$.

Now $$\frac{MD}{MA}=\frac{\sin MAD}{\sin MDA}=\frac{\sin\theta}{\cos\alpha};\quad \therefore\ MD=\frac{x\sin\theta}{\cos\alpha}.$$

$$EM=FL=FA-AL=2c\sin\alpha-AL;$$

$$\frac{AL}{AM}=\frac{\sin AML}{\sin ALM}=\frac{\sin(\pi-\theta-2\alpha)}{\sin\left(\frac{\pi}{2}+\alpha\right)};$$

$$\therefore\ AL=\frac{x\sin(\theta+2\alpha)}{\cos\alpha},$$

$$\therefore\ EM=2c\sin\alpha-\frac{x\sin(\theta+2\alpha)}{\cos\alpha}.$$

But, from a property of the circle, $MP^2=EM\,.\,MD$;

$$\therefore\ y^2=\frac{x\sin\theta}{\cos\alpha}\left\{2c\sin\alpha-\frac{x\sin(\theta+2\alpha)}{\cos\alpha}\right\}.$$

If we compare this equation with that in Art. 282, we see that the section is an ellipse, hyperbola, or parabola, according as $-\frac{\sin\theta\sin(\theta+2\alpha)}{\cos^2\alpha}$ is negative, positive, or zero, that is, according as $\theta+2\alpha$ is less that π, greater than π, or equal to π.

Hence if AM is parallel to OB the section is a parabola, if AM produced through M meets OB the section is an ellipse, if AM produced through A meets OB produced through O the section is an hyperbola.

If $c=0$ the section is a point if $\theta+2\alpha$ is less than π, two straight lines if $\theta+2\alpha$ is greater than π, and one straight line if $\theta+2\alpha=\pi$. The section is also a straight line whatever c may be, if $\theta=0$ or π.

The equation above obtained may be written

$$y^2=\frac{\sin\theta\sin(\theta+2\alpha)}{\cos^2\alpha}\left\{\frac{2c\sin\alpha\cos\alpha}{\sin(\theta+2\alpha)}x-x^2\right\}$$

suppose $\theta+2\alpha$ to be less than π, so that the curve is an ellipse; then by comparing this equation with the equation $y^2=\frac{b^2}{a^2}(2ax-x^2)$, we have

$$2a=\frac{2c\sin\alpha\cos\alpha}{\sin(\theta+2\alpha)},\quad \frac{b^2}{a^2}=\frac{\sin\theta\sin(\theta+2\alpha)}{\cos^2\alpha}.$$

Thus $$2a=\frac{c\sin 2\alpha}{\sin(\theta+2\alpha)},\qquad b^2=\frac{c^2\sin^2\alpha\sin\theta}{\sin(\theta+2\alpha)}.$$

Also $$e^2=1-\frac{b^2}{a^2}=\frac{\cos^2\alpha-\{\sin^2(\theta+\alpha)-\sin^2\alpha\}}{\cos^2\alpha}=\frac{\cos^2(\theta+\alpha)}{\cos^2\alpha}.$$

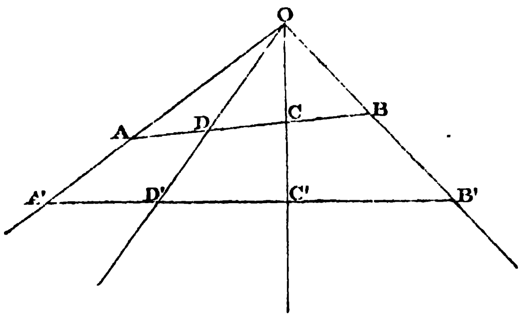
If we suppose in the figure of Art. 324 that AM is produced to meet the cone again in A', then $2a=AA'$, as might have been anticipated; also b may be shewn to be a mean proportional between the perpendiculars from A and A' on the axis OH. Similar results may be obtained when the curve is an hyperbola.

Anharmonic Ratio and Harmonic Pencil.

325. We will now give a short account of anharmonic ratios and harmonic pencils, which are often used in investigating and enunciating properties of the conic sections.

Let there be four straight lines meeting in a point; then if *any* straight line $ADCB$ be drawn across the system,

$$\frac{AB}{AC} \div \frac{DB}{DC} \text{ will be a constant ratio.}$$



Suppose O the point where the lines meet; then

$$\frac{AB}{AO} = \frac{\sin AOB}{\sin ABO},$$

$$\frac{AC}{AO} = \frac{\sin AOC}{\sin ACO};$$

$$\frac{AB}{AC} = \frac{\sin AOB}{\sin AOC} \cdot \frac{\sin ACO}{\sin ABO}.$$

Similarly $$\frac{DB}{DC} = \frac{\sin DOB}{\sin DOC} \cdot \frac{\sin DCO}{\sin DBO};$$

$$\therefore \frac{AB}{AC} \div \frac{DB}{DC} = \frac{\sin AOB}{\sin AOC} \div \frac{\sin DOB}{\sin DOC}.$$

Now suppose any other straight line $A'D'C'B'$ drawn across the system, then since AOB and $A'OB'$ are the same angle, and so on for the other angles, we have

$$\frac{AB}{AC} \div \frac{DB}{DC} = \frac{A'B'}{A'C'} \div \frac{D'B'}{D'C'},$$

which proves the proposition.

Similarly we can prove that each of the following is a constant ratio,

$$\frac{AB}{AD} \div \frac{CB}{CD} \text{ and } \frac{AC}{AD} \div \frac{BC}{BD}.$$

326. DEFS. Any four lines meeting in a point form a *pencil*.

A straight line drawn across a pencil is called a *transversal*.

Any one of the constant ratios $\frac{AB}{AC} \div \frac{DB}{DC}$, $\frac{AB}{AD} \div \frac{CB}{CD}$, $\frac{AC}{AD} \div \frac{BC}{BD}$ is called *an anharmonic ratio* of the pencil.

The pencil is called *harmonic* if $AB.DC = AD.BC$, that is, if the rectangle formed by the *whole* line (AB) and the middle part (DC) is equal to the rectangle of the other two parts (AD), (BC).

327. The *harmonic* pencil is so called because it divides any transversal harmonically. For since $AB.DC = AD.BC$,

$$\frac{AB}{AD} :: \frac{BC}{DC},$$

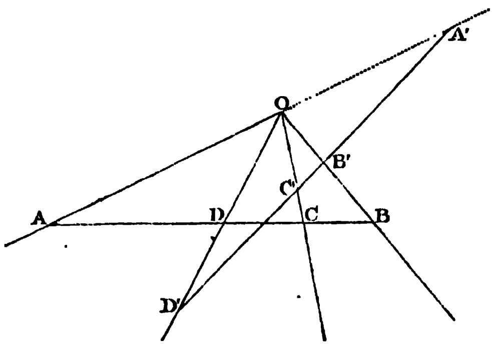
that is, if we call AB, AC, AD, the first, second, and third quantities respectively, the first is to the third as the difference of the first and second is to the difference of the second and third.

When the pencil is harmonic *one* of the three constant ratios of the pencil is equal to unity.

We shall sometimes select one of the anharmonic ratios of a pencil, and confine our attention to it, and shall then speak of the selected ratio as *the* anharmonic ratio of the pencil.

328. Suppose OA, OB, OC, OD form an harmonic pencil; if we take any new origin O', and join $O'A$, $O'B$, $O'C$, $O'D$, these four lines form a new harmonic pencil; for the transversal $ABCD$ is cut harmonically.

329. The anharmonic ratio of a pencil is not altered if the transversal meet the lines of the pencils *produced*, instead of the lines themselves.



Suppose OA, OB, OC, OD to be a pencil, and let a transversal $A'B'C'D'$ meet three lines of the pencil, and the fourth AO produced in A'. The angles $A'OB'$, AOB are supplemental; and so are AOD, $A'OD'$; and so on. Hence any anharmonic ratio formed on $ABCD$ is equal to the corresponding ratio formed on $A'B'C'D'$.

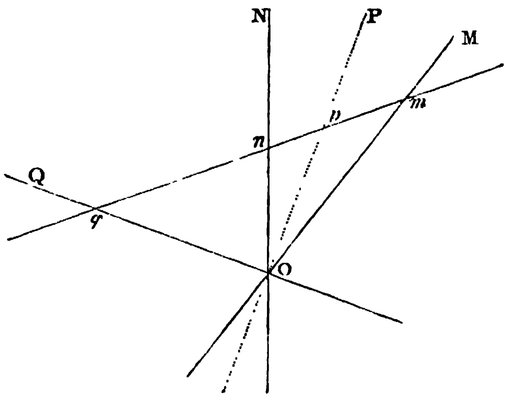
330. Suppose $AB.CD = AD.BC$, so that OA, OB, OC, OD form an harmonic pencil. By the last proposition

$$\frac{A'B'}{A'D'} \div \frac{C'B'}{C'D'} = \frac{AB}{AD} \div \frac{CB}{CD} = 1;$$

$\therefore$ OA', OB', OC', OD' form an harmonic pencil.

Similarly OC', OB', OA', and DO produced through O will form an harmonic pencil. Thus from one harmonic pencil by producing the lines through the vertex, we can derive four other harmonic pencils.

331. The lines whose equations are $\alpha=0$, $\beta=0$, $\alpha-k\beta=0$, $\alpha+k\beta=0$ form an *harmonic* pencil.



Let OM be the line $\alpha=0$,

ON .... $\beta=0$,

OP .... $\alpha-k\beta=0$,

OQ .... $\alpha+k\beta=0$.

Let a transversal meet the pencil in $mpnq$; then (Art. 72)

$$\frac{\sin POM}{\sin PON}=k=\frac{\sin QOM}{\sin QON};$$

$$\therefore \frac{\sin POM}{\sin PON}\cdot\frac{\sin QON}{\sin QOM}=1;$$

$$\therefore \text{(as in Art. 325) } \frac{pm}{pn}\cdot\frac{qn}{qm}=1;$$

$$\therefore pm\,.\,qn=pn\,.\,qm.$$

The same result will follow if we draw the transversal in a different position. The harmonic pencil is so formed that its outside lines are always one of the two $\alpha = 0$ and $\beta = 0$, and one of the two $\alpha - k\beta = 0$ and $\alpha + k\beta = 0$.

332. The anharmonic ratio of the four lines $\alpha = 0$, $\beta = 0$, $\alpha - k\beta = 0$, $\alpha + k'\beta = 0$, is $\frac{k}{k'}$.

For as in the preceding article we have

$$\frac{\sin POM}{\sin PON} = k, \quad \frac{\sin QOM}{\sin QON} = k';$$

therefore, by Art. 326, $\frac{k}{k'}$ expresses the anharmonic ratio.

333. Article 331 will also hold if the equations to the lines be $u = 0$, $v = 0$, $u - kv = 0$, and $u + kv = 0$. For, by Art. 57, we have $u = \lambda\alpha$, $v = \mu\beta$, where λ and μ are constant quantities; hence the equations $u - kv = 0$ and $u + kv = 0$ may be written $\lambda\alpha - k\mu\beta = 0$ and $\lambda\alpha + k\mu\beta = 0$, or $\alpha - k'\beta = 0$ and $\alpha + k'\beta = 0$, where $k' = \frac{k\mu}{\lambda}$. Hence Article 331 becomes immediately applicable.

334. The four lines EB, EC, EG, EF, in Art. 76, form an harmonic pencil; for their equations are

$$u = 0, \; w = 0, \; lu - nw = 0, \; lu + nw = 0.$$

By symmetry FB, FA, FG, FE, will also form an harmonic pencil.

Also GD, GC, GF, GE form an harmonic pencil, for their equations are respectively

$$lu - mv = 0, \quad mv - nw = 0, \quad lu - mv - (mv - nw) = 0,$$
$$lu - mv + mv - nw = 0.$$

335. A straight line drawn through the intersection of two tangents to a conic section is divided harmonically by the curve and the chord of contact.

Refer the curve to the tangents as axes; its equation will be of the form (Art. 293)

$$\left(\frac{x}{h}+\frac{y}{k}-1\right)^2+\mu xy=0 \text{ .... } (1).$$

Suppose a straight line drawn through the origin, and let its equation be (Art. 27)

$$\frac{x}{l}=\frac{y}{m}=r \text{ .... } (2).$$

Thus the distances from the origin of the points of intersection of (1) and (2) will be the values of r found from the equation

$$\left(\frac{lr}{h}+\frac{mr}{k}-1\right)^2+\mu lmr^2=0,$$

or

$$\left(\frac{l}{h}+\frac{m}{k}-\frac{1}{r}\right)^2+\mu lm=0 \text{ .... } (3).$$

If r' and r'' be the roots of the equation, we have

$$\frac{1}{r'}+\frac{1}{r''}=2\left(\frac{l}{h}+\frac{m}{k}\right) \text{ .... } (4).$$

Also the equation to the chord of contact is

$$\frac{x}{h}+\frac{y}{k}-1=0 \text{ .... } (5).$$

Hence for the distance (r_1) of the point of intersection of (2) and (5) from the origin, we have the equation

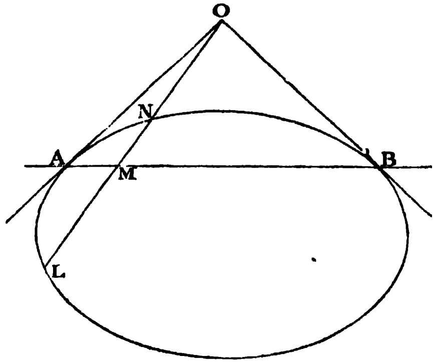
$$\frac{lr_1}{h}+\frac{mr_1}{k}=1, \text{ or } \frac{1}{r_1}=\frac{l}{h}+\frac{m}{k} \text{ .... } (6).$$

From (4) and (6) we have

$$\frac{2}{r_1}=\frac{1}{r'}+\frac{1}{r''},$$

thus r_1 is an harmonic mean between r' and r''.

Since $LMNO$ is divided harmonically, if from any point in AB we draw lines to L, N, and O, these lines with AB form



an harmonic pencil. A particular case is that in which the point in AB is the intersection of the tangents at N and L, which we know will meet on AB. (See Arts. 103, 186.)

336. Let A, B, C, D be four points on a conic section, and P any fifth point. Let α denote the perpendicular from P on AB, β the perpendicular from the same point on BC, γ on CD, δ on DA. Then by Art. 317 we know that wherever P may be, $\alpha\gamma$ bears a constant ratio to $\beta\delta$. Now $AB \, . \, \alpha =$ twice the area of the triangle PAB

$$= PA \, . \, PB \, . \sin APB;$$

$$\alpha = \frac{PA \, . \, PB \, . \sin APB}{AB}.$$

Similar values may be found for β, γ, δ. Thus

$$\frac{PA \, . \, PB \, . \, PC \, . \, PD}{AB \, . \, CD} \sin APB \, . \sin CPD$$

bears a constant ratio to

$$\frac{PA \, . \, PB \, . \, PC \, . \, PD}{BC \, . \, AD} \sin BPC \, . \sin DPA:$$

$\therefore \dfrac{\sin APB \, . \sin CPD}{\sin BPC \, . \sin DPA}$ is constant, that is, the pencil drawn from any point P to the four points A, B, C, D, has a constant anharmonic ratio.

ANSWERS TO THE EXAMPLES.

CHAPTER I.

8. The co-ordinates of D are $\frac{1}{2}(x_1 + x_2)$ and $\frac{1}{2}(y_1 + y_2)$. The co-ordinates of G are $\frac{1}{3}(x_1 + x_2 + x_3)$ and $\frac{1}{3}(y_1 + y_2 + y_3)$.

10. Let r and θ be the polar co-ordinates of C. Then the angle $AOC =$ the angle BOC; that is, $\theta - \theta_1 = \theta_2 - \theta$; $\therefore\ \theta = \frac{1}{2}(\theta_1 + \theta_2)$.

Again, from the known expression for the area of a triangle (see *Trigonometry*, Chapter XVI.), we have

$$\text{triangle } AOB = \tfrac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1),$$
$$\text{triangle } AOC = \tfrac{1}{2} r_1 r \sin(\theta - \theta_1),$$
$$\text{triangle } BOC = \tfrac{1}{2} r_2 r \sin(\theta_2 - \theta).$$

Thus $$r_1 r_2 \sin(\theta_2 - \theta_1) = r_1 r \sin(\theta - \theta_1) + r_2 r \sin(\theta_2 - \theta)$$
$$= r(r_1 + r_2)\sin\tfrac{1}{2}(\theta_2 - \theta_1);$$
$$\therefore\ r(r_1 + r_2) = 2r_1 r_2 \cos\tfrac{1}{2}(\theta_2 - \theta_1).$$

CHAPTER III.

1. (1) $y + 2x = 1$. (2) $x = 2$. (3) $y = x$. (4) $x = 0$.
2. $y - 4 = -3(x - 4)$, $y - 4 = \frac{1}{3}(x - 4)$.
3. $y - 1 = (\sqrt{3} - 2)x$, $y - 1 = -(\sqrt{3} + 2)x$.
4. $y = x$, $y = -x$. 5. $y = \dfrac{1}{\sqrt{3}}x$, $x = 0$.
6. 90°, $x = -\frac{1}{2}$, $y = \frac{3}{2}$. 7. 60°. 8. 45°.
9. $y = \pm(x - a)$. 10. $y = x$. 11. $2\sqrt{2}$.

12. $\frac{ab}{\sqrt{(a^2+b^2)}}$. 13. $x=y=\frac{ab}{a+b}$. 14. $\frac{x}{a^2}-\frac{y}{b^2}=\frac{1}{a}-\frac{1}{b}$.

15. (1) The origin. (2) Two straight lines, $y=x$ and $y=-x$. (3) Two straight lines, $x=0$ and $x+y=0$. (4) The axes. (5) Impossible. (6) Two straight lines, $x=0$ and $y=a$. 16. (1) Two straight lines, $x=a$ and $y=b$. (2) The point (a, b). (3) The point $(0, a)$. 17. The lines $y=x$ and $y=3x$. 19. $4y=5x$, and $3y+2x-20=0$. 20. Let a be the length of the side of the hexagon; the equations are to AB, $y=0$; AC, $y\sqrt{3}=x$; AD, $y=x\sqrt{3}$; AE, $x=0$; AF, $y+x\sqrt{3}=0$; BC, $y=\sqrt{3}(x-a)$; BD, $x=a$; BE, $y+\sqrt{3}(x-a)=0$; BF, $y\sqrt{3}+x-a=0$; CD, $y+x\sqrt{3}=2a\sqrt{3}$; CE, $y\sqrt{3}+x=3a$; CF, $2y=a\sqrt{3}$; DE, $y=\sqrt{3}a$; DF, $y\sqrt{3}-x=2a$; EF, $y-x\sqrt{3}=a\sqrt{3}$. 21. If (x_1, y_1), (x_2, y_2), (x_3, y_3) be the angular points, the co-ordinates of the point midway between the first and second are $\frac{x_1+x_2}{2}$, $\frac{y_1+y_2}{2}$; similarly the co-ordinates of the point midway between the second and third points are known; and then the required equation can be found by Art. 35. 22. $\frac{m^2+1}{m^2-1}\tan\omega$. 24. $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{a}=\frac{y}{b}$; tangent of the angle between them $\frac{2ab\sin\omega}{a^2-b^2}$. 29. The points whose abscissæ are $a+\frac{a}{b}\sqrt{(a^2+b^2)}$ and $a-\frac{a}{b}\sqrt{(a^2+b^2)}$.

31. $\frac{\sqrt{(B^2-4AC)}}{A+C}$. 35. $90°$. 36. $F(\theta)=0$ gives a system of lines through the origin; $\sin 3\theta=0$ gives the three lines $y=0$, $y=x\sqrt{3}$, $y=-x\sqrt{3}$. 40. The second pair of lines bisect the angles included by the first pair. 44. Let ABC be the triangle; take A for the origin and lines through A parallel to the two given lines as axes; let x_1, y_1 be the co-ordinates of B, and x_2, y_2 those of C. Then it may be shewn that the equations to the three diagonals mentioned are

$$y-y_2=\frac{y_1-y_2}{x_2-x_1}(x-x_1),\quad y-y_2=-\frac{y_2}{x_2}x,\quad y-y_1=-\frac{y_1}{x_1}x;$$

from these equations it may be shewn that the three diagonals meet in a point. 45. Take O as origin and use polar equations

to the given fixed straight lines. 46. Let x_1 be the abscissa of the point of intersection of the two lines; then the area of the triangle is $\frac{1}{2}(c_2 - c_1)x_1$. 47. This may be solved by Art. 11. Or we may use the result of the preceding question; for by drawing a figure we shall obtain *three* triangles to which the preceding question applies, and the required area is the difference between two of these triangles and the third. The result is

$$\pm\left\{\frac{(c_3-c_2)^2}{2(m_3-m_2)}+\frac{(c_1-c_3)^2}{2(m_1-m_3)}+\frac{(c_2-c_1)^2}{2(m_2-m_1)}\right\},$$

which may also be written thus

$$\mp\frac{\{c_1(m_3-m_2)+c_2(m_1-m_3)+c_3(m_2-m_1)\}^2}{2(m_3-m_2)(m_1-m_3)(m_2-m_1)}.$$

That sign should be taken which gives a positive result.

CHAPTER IV.

1. $\frac{x}{a}+\frac{y}{b}=\frac{x}{a'}+\frac{y}{b'}$.

7. Since the required line is parallel to the line considered in Example 5, we may assume for its equation

$$\alpha\cos A-\beta\cos B+k=0,$$

where k is some *constant* to be determined. Now at the middle point of AB, we have

$$-\alpha=\frac{c}{2}\sin B,\quad -\beta=\frac{c}{2}\sin A;$$

therefore $-\frac{c}{2}\sin B\cos A+\frac{c}{2}\sin A\cos B+k=0$

thus k is determined. $\therefore k = \frac{c}{2}(\sin(B-A))$

13. $(mn'-m'n)u+(nl'-n'l)v+(lm'-l'm)w=0$.

14. $ab(u-v)+c(b+a)w=0$.

15. Assume for the required equation $l\alpha+m\beta+n\gamma=0$; at the centre of the inscribed circle $\alpha=\beta=\gamma$; thus $l+m+n=0$; at the centre of the circumscribed circle α, β, γ are proportional respec-

tively to $\cos A$, $\cos B$, $\cos C$; thus $l \cos A + m \cos B + n \cos C = 0$. Hence the required result may be obtained.

18. To CP, $2mv - nw = 0$; to DP, $2lu - 2mv + nw = 0$;
to AQ, $lu - 2mv + 2nw = 0$; to BQ, $lu - 2mv = 0$.

23. It may be shewn that if $u = 0$, $v = 0$, $w = 0$ denote the sides of the triangle, the lines AP, BP, CP may be denoted by $mv - nw = 0$, $nw - lu = 0$, and $lu - mv = 0$ respectively; then the equations to the other lines can easily be expressed.

24. Take $\alpha = 0$, $\beta = 0$, $\gamma = 0$ to represent the sides of the triangle $A'B'C'$; then the equations to BC, CA, AB will be respectively $\beta + \gamma = 0$, $\gamma + \alpha = 0$, $\alpha + \beta = 0$. Then the equation to AA' will be $\beta - \gamma = 0$, so that AA' is perpendicular to BC.

25. The equation to OO' is $\beta - \gamma = 0$; take $\beta - \gamma - \lambda\alpha = 0$ for the equation to the line drawn through D. Then it will be found that the equation to OF is $\beta - \gamma - \lambda(\alpha - \gamma) = 0$, and that the equation to $O'E$ is $\beta - \gamma - \lambda(\alpha + \beta) = 0$. Thus at the point P we have $\beta = -\gamma$. The same relation holds at the point Q.

CHAPTER V.

1. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$. 3. $y'^2 = c\sqrt{2x' - \frac{c^2}{2}}$.

4. $y'^2 \sin^2\alpha = 4ax'$. 6. By Art. 83, we have

$$m = \frac{\sin(\omega - \alpha)}{\sin\omega}, \quad n = \frac{\sin(\omega - \beta)}{\sin\omega}, \quad m' = \frac{\sin\alpha}{\sin\omega}, \quad n' = \frac{\sin\beta}{\sin\omega}.$$

CHAPTER VI.

1. (1) Co-ordinates of the centre 2 and -2, radius 3.
(2) Co-ordinates of the centre -3 and $\frac{3}{2}$, radius $\frac{7}{2}$.

2. The first line meets the circle at the points $(-4, 3)$ and $(3, -4)$; the second at the points $(0, -5)$ and $(-5, 0)$; the third *touches* it at the point $(-4, -3)$.

5. $x^2 - x(x' + x'') + y^2 - y(y' + y'') + x'x'' + y'y'' = 0$.

8. For determining the abscissæ of the points of intersection

we have $x^2\left(1 - \frac{k^2}{h^2}\right) + \frac{2k}{h}(b-k)x - 2ax + k^2 - 2bk = 0$; if the line *touches* the circle we must have $(kb - ha)^2 + 2kh(ka + hb) = h^2k^2$.
9. $2y + 3x = 0$. 14. $x^2 + y^2 - xy - hx - ky = 0$.
15. Inclination of axes 120°; co-ordinates of the centre each $= h$; radius $= h$. 16. Inclination of axes 60°; co-ordinates of the centre each $= \frac{h}{3}$; radius $= \frac{h}{\sqrt{3}}$. 17. $x^2 + y^2 + xy\sqrt{2} - 9 = 0$.

18. $x^2 + y^2 + xy + x + y - 1 = 0$. 19. $\frac{\sqrt{(h^2 + k^2 - 2hk\cos\omega)}}{2\sin\omega}$

23. $x^2 + y^2 = a\left(x + \frac{y}{\sqrt{3}}\right)$; $r = \frac{2a}{\sqrt{3}}\cos\left(\theta - \frac{\pi}{6}\right)$. 27. A circle.
28. Use the equation in question 26. 29. Using polar co-ordinates, we have

$$r + \sqrt{(r^2 + a^2 - 2ra\cos\theta)} = \sqrt{\left\{r^2 + a^2 - 2ra\cos\left(\frac{\pi}{3} - \right.\right.}$$

reduce and we get $\left\{\sqrt{3}r - 2a\cos\left(\theta - \frac{\pi}{6}\right)\right\}^2 = 0$; thus the locus is the circle circumscribing the triangle.

30. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \ldots = \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \ldots$
and $\sin 2\alpha + \sin 2\beta + \sin 2\gamma + \ldots = 0$.

32. If the perpendiculars are both on the same side of the line the locus is a circle; if on different sides the locus consists of two straight lines. 33. A circle. 34. A circle. 36. Solve the quadratic in r; it will be found that $r = 2a\cos\theta$ or $-a\sec\theta$; thus the locus consists of a straight line and a circle.

38. Take the extremity of the diameter as the pole; it will follow from Example 37, that the tangent at P is represented by the equation $2a\cos^2\alpha = r\cos(2\alpha - \theta)$, and the tangent at Q by the equation $2c\cos^2\beta = r\cos(2\beta - \theta)$. These tangents meet at T, so at that point we have

$$\frac{\cos(2\alpha - \theta)}{\cos^2\alpha} = \frac{\cos(2\beta - \theta)}{\cos^2\beta};$$

from this we shall find $\tan\theta = \frac{\sin(\beta + \alpha)}{2\cos\beta\cos\alpha}$,

so that if C be the centre of the circle $Ct = \dfrac{c \sin(\beta + \alpha)}{2 \cos\beta \cos\alpha}$.

Hence we can shew that $Cq - Ct = Ct - Cp$.

CHAPTER VII.

4. $x = y$, and $x + y = \dfrac{3ab}{a + b}$.

5. Let $y = mx$ be the equation to one line; then

$$\delta = \frac{y_1 - mx_1}{\pm\sqrt{(1 + m^2)}}; \qquad \therefore\ \delta^2 (1 + m^2) = (y_1 - mx_1)^2;$$

this is a quadratic for finding m, and we may replace m by $\dfrac{y}{x}$.

6. $A^2c^2 + C^2a^2 + B^2ac + ACb^2 - 2ACac - Bb\,(Ac + Ca) = 0$.

CHAPTER VIII.

1. $y = 2x$. 2. $y^2 = 5ax - x^2$. 3. The locus consists of two parabolas of which the centre of the circle is the common focus, and the directrices are the two tangents to the circle which are parallel to the fixed diameter. 4. The second curve is a parabola having its axis coinciding with the negative part of the axis of y; the curves intersect at the origin and at the point $x = 4a$, $y = -4a$. 5. $y = x + a$. 6. $\tan^{-1} \frac{1}{3}$. 7. $y + x = 3a$. 8. At the point $(9a, -6a)$; length $8a\sqrt{2}$. 9. $y = 2a\sqrt{3}$, $x = 3a$. 11. The abscissa of the required point is 0 or $3a$. 13. The curve is a parabola having its axis parallel to that of y, and its vertex at the point $x = \frac{1}{2}$, $y = \frac{1}{4}$. The line is a tangent at the point $x = 1$, $y = 0$. 20. Abscissa of required point is $\dfrac{1}{4a}\left(\dfrac{8a^2}{y'} + y'\right)^2$, ordinate $-\left(\dfrac{8a^2}{y'} + y'\right)$; length of chord $\dfrac{2}{y'^2}(4a^2 + y'^2)^{\frac{3}{2}}$. 22. Locus of Q, $x = -2a$. Locus of Q', $x^3 = ay^2$. 23. Refer the parabola to PT and the diameter at P as axes. See Art. 151. 25. See Art. 155. 27. Transform equation (1) of Art. 125 to polar co-ordinates, and we shall deduce $r = 2a\dfrac{\cos\theta \pm \sqrt{(\cos 2\theta)}}{\sin^2\theta}$.

28. Use the result of the preceding example.

29. $r = 2a \dfrac{\sin\theta \pm \sqrt{(-\cos 2\theta)}}{\cos^2\theta}$. 30. The locus is a parabola; see Art. 147. 32. $\sqrt{x} + \sqrt{y} = \sqrt{(a2\sqrt{2})}$.
33. $(y' - x')^2 - 8ax'\sqrt{2} = 0$. 34. $x^2 + y^2 - x(a + x') - yy' + ax' = 0$.
37. Use the result of example 5, Chap. VI. 41. The equation to one tangent can be written $y = m(x + a) + \dfrac{a}{m}$, (see example 40), and that to the other $y = -\dfrac{1}{m}(x + a') - a'm$. By eliminating m we have for the required locus $x + a + a' = 0$. 42. Take for the equation to the chord $y = mx + n$; then to find the abscissa of the middle point of the chord we must take *half the sum of the roots* of the equation $(mx + n)^2 = 4ax$; so that the abscissa is $\dfrac{2a - mn}{m^2}$. Now since the chord *touches* the parabola $y^2 = 8a(x - c)$ the equation $(mx + n)^2 = 8a(x - c)$ must have *equal* roots; by means of this condition it can be shewn that $\dfrac{2a - mn}{m^2} = c$. 44. Form the equation which determines the ordinate of a point in a parabola such that the normal shall pass through a given point; the equation will be a cubic. 45. The tangents of the inclinations to the axis of x of the three normals that can be drawn through a point (x, y) are determined by the equation $m^3 + m\left(2 - \dfrac{x}{a}\right) + \dfrac{y}{a} = 0$. See Art. 135. Suppose m_1, m_2, m_3 the roots of this cubic, then by the theory of equations $m_1 + m_2 + m_3 = 0$, $m_2m_3 + m_3m_1 + m_1m_2 = 2 - \dfrac{x}{a}$, $m_1m_2m_3 = -\dfrac{y}{a}$; if two of the normals are at right angles we may put $m_2m_3 = -1$; from these equations by eliminating m_1, m_2, and m_3, we find $y^2 = a(x - 3a)$. 46. By the breadth is meant the distance between the two tangents which are parallel to PQ. 47. $\dfrac{k^2 - 4ah}{\sqrt{(k^2 + 4a^2)}}$.
55. The equation $y = mx + \dfrac{a}{m}$ represents a tangent to the parabola; if this passes through the point (h, k) we have $k = mh + \dfrac{a}{m}$; also

$m=\dfrac{y-k}{x-h}$, where (x, y) is any point on the tangent; thus $k-\dfrac{y-k}{x-h}h=\dfrac{a(x-h)}{y-k}$; this will give the first form of the equation. The second form may be deduced from the first; the student will see hereafter what suggested the second form; see Arts. 321 and 322.

56. The equation $y^2=4ax$ represents the parabola; and the equation $ky-2ax=2ah$ represents the chord of contact; hence the equation $4ax(ky-2ax)=2ahy^2$ represents *some locus* passing through the intersection of the parabola and chord; then see Art. 61.

CHAPTER IX.

1. $\dfrac{1}{\sqrt{3}}$. 2. $y+ex=a$; the intercept on the axis of $x=\dfrac{a}{e}$; and the intercept on the axis of $y=a$. 3. $y+ae^2=\dfrac{x}{e}$.

4. The excentricity is determined by $e^4+e^2=1$. 5. $y=\dfrac{b}{a}(x+a)$; $y=\dfrac{b^2x}{a^2e}$; the lines are parallel if $2e^2=1$. 6. $y=\dfrac{b}{ae}(x-ae)$; the abscissa of the point of intersection is $\dfrac{2ae}{1+e^2}$.

7. $y=-(1+e)(x-a)$; $\tan^{-1}\dfrac{1}{1+e+e^2}$. 8. $\dfrac{2a^2e-ax'(1+e^2)}{a(1+e^2)-2ex'}$.

9. The co-ordinates of the point are $x=\dfrac{a^2}{\sqrt{(a^2+b^2)}}$, $y=\dfrac{b^2}{\sqrt{(a^2+b^2)}}$.

10. The co-ordinates of the point are $x=\dfrac{a}{\sqrt{2}}$, $y=\dfrac{b}{\sqrt{2}}$.

19. It will be found that the circle falls entirely without the ellipse if the inclination of the two parallel straight lines to the major axis be greater than $\tan^{-1}\dfrac{ae}{b}$. 22. $\dfrac{x}{a}\cos\phi+\dfrac{y}{b}\sin\phi=1$.

25. The co-ordinates of the required point are $x=\dfrac{ae(b-a)}{b-ae^2}$, $y=\dfrac{ab(1-e^2)}{b-ae^2}$; the lines are parallel when $e^4+e^2=1$.

28. $x^2+y^2-x(ae+x')-yy'+aex'=0$. 30. If the point (h, k) be

between the directrices, the sum of the perpendiculars is $\dfrac{2a^2b^2}{\sqrt{(a^4k^2+b^4h^2)}}$; if the point (h, k) be *not* between the directrices, the sum of the perpendiculars is $\pm\dfrac{2ab^2he}{\sqrt{(a^4k^2+b^4h^2)}}$, the upper or lower sign being taken according as h is positive or negative. 31. A circle having its centre at the centre of the ellipse and radius $= a+b$.

32. $y = \pm x \pm \sqrt{(a^2+b^2)}$. See Art. 171. 34. Locus is the circle $x^2+y^2 = a^2+b^2$; this may be deduced from the second part of example 33. 35. See remark on Ex. 55 of Chap. VIII. 42. The first part of this example may be solved by finding the equation to the line passing through the points of intersection of the two ellipses. 45. $x^2+y^2 = (a^2+b^2)^{\frac{1}{2}}(x+y)$. 46. Let h, k be the co-ordinates of an external point; the equation to the corresponding chord of contact is $a^2ky + b^2hx = a^2b^2$; the equation to the line through (h, k) perpendicular to the chord is $(y-k)\,b^2h = a^2k\,(x-h)$. We require that the latter line shall be a tangent to the ellipse; the necessary condition may be found by comparing this equation with the equation $y = mx + \sqrt{(m^2a^2+b^2)}$; thus we shall obtain for the condition $k^2a^6 + h^2b^6 = h^2k^2\,(a^2-b^2)^2$. 48. $a^2(y^2+2yk) + b^2(x^2+2xh) = 0$. 52. An ellipse. 53. The locus is an ellipse; if A be the origin, AB the axis of x, each of the co-ordinates of the focus is equal to half the radius of the circle.

54. $\dfrac{e^2xy}{b^2}$. 55. Put $a\cos\phi$ for x and $b\sin\phi$ for y in the preceding result (Art. 168); then the greatest value is $\dfrac{e^2a}{2b}$.

57. Let P denote a point on the ellipse, and Q the centre of the circle inscribed in the triangle SPH; then if y' be the ordinate of P it may be shewn that the radius of the circle which $= \dfrac{\text{area of triangle } SPH}{\text{semiperimeter of triangle}} = \dfrac{ey'}{1+e}$; this is the ordinate of Q. Let x' be the abscissa of P, then it may be shewn that the abscissa of Q is ex'; thus it will be found that the required locus is an ellipse. 58. Find the point in which SZ meets the normal at P; also find the point in which HZ' meets the normal at P; it will then appear that the points coincide.

CHAPTER X.

1. $xb(bx' - ay') + ya(ay' + bx') = a^2b^2$. 2. Refer the ellipse to the diameter and its conjugate as axes. 3. See Art. 11. 8. $r(a^2\sin^2\theta + b^2\cos^2\theta) = 2ab^2\cos\theta$. 9 and 10. Use the result of 8. 12. Result the same as that in Ex. 11. 13. They intersect when $\theta = 0$ and when $\theta = \frac{\pi}{2}$. 14. The equations to the tangents at the ends of the latera recta are (Art. 205)

$$r(e\cos\theta + \sin\theta) = a(1 - e^2); \qquad r(\sin\theta - e\cos\theta) = a(1 + e^2);$$

$$r(e\cos\theta - \sin\theta) = a(1 - e^2); \qquad r(\sin\theta + e\cos\theta) = -a(1 + e^2).$$

The equations to the tangents at the ends of the minor axis are $r\sin\theta = b$; $r\sin\theta = -b$. 15. A straight line through S. See Art. 205. 17. $\cos\theta = -\frac{e + e'}{1 + ee'}$, $r = a(1 + ee')$. 18. Between $\frac{b}{a}$ and $\frac{a}{b}$. 20. See Art. 208. 22. The sine of the angle between the radius vector from the centre and the tangent is $\frac{p}{r}$, where $p^2(a^2 + b^2 - r^2) = a^2b^2$ by Art. 196; then the least value of $\frac{p^2}{r^2}$ may be shewn to be when $2r^2 = a^2 + b^2$. 29. It may be shewn that the axis of the parabola must coincide with *one* of the axes of the ellipse, hence the latus rectum will be either $\frac{2a^2}{\sqrt{(a^2 + b^2)}}$ or $\frac{2b^2}{\sqrt{(a^2 + b^2)}}$. 31. An ellipse. 32. An ellipse. 35. Use the polar equations to PQ and pq; see Art. 205. 38. Two of the sides of the parallelogram are determined by the equations $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = \pm 1$, and the other two by the equations $\frac{x}{a}\cos\phi' + \frac{y}{b}\sin\phi' = \pm 1$; see example 22 of Chap. IX. It may be shewn that the diagonals of the parallelogram intersect at the centre of the ellipse; then if the centre of the ellipse be joined with two adjacent corners of the parallelogram the triangle thus formed is one fourth of the parallelogram; and

the area of the triangle is known by example 7 of Chap. I. 41. The abscissa is $\frac{bx'-ay'}{b}$, and the ordinate $\frac{ay'+bx'}{a}$. 42. The co-ordinates of the intersection of the tangents are found in Ex. 41; call them h and k, then use the second form given in Ex. 35 of Chap. IX. 44. The greatest value may be found by substituting for x' and y' their values from Art. 168; it is $ab(\sqrt{2}-1)$. 47. An ellipse. 48. An ellipse referred to its *equal* conjugate diameters. 51. This may be solved by means of Ex. 50. Or we may take the usual axes, then if x', y' be the co-ordinates of P those of M will be $\frac{a(ax'+by')}{a^2+b^2}$ and $\frac{b(ax'+by')}{a^2+b^2}$; those of N will be $\frac{a(ax'-by')}{a^2+b^2}$ and $\frac{b(by'-ax')}{a^2+b^2}$. Hence the solution can be completed.

CHAPTER XI.

1. $y^2-3x^2=-3a^2$. 2. A straight line.

CHAPTER XII.

3. Let a line be drawn through the focus meeting the hyperbola in P and p and the asymptotes in Q and q; then it may be shewn that $Pp=\frac{2a(e^2-1)}{1-e^2\cos^2\theta}=\frac{2a\sin^2\alpha}{\cos^2\alpha-\cos^2\theta}$, $Qq=\frac{2a\sin\alpha\sin\theta}{\cos^2\alpha-\cos^2\theta}$, and the required length is half the difference of Pp and Qq. 4. Take the centre of the circle as the origin, AB as the axis of x, and a diameter parallel to PQ as the axis of y; then the locus is given by the equation $y^2=x^2-a^2$, and is therefore a rectangular hyperbola referred to conjugate diameters. 9. By example 53 of Chapter IX. we shall obtain $\tan\alpha=\frac{\sqrt{(k^2-4ah)}}{h+a}$; $\therefore (h+a)^2\tan^2\alpha=k^2-4ah$; $\therefore (h+a)^2\sec^2\alpha=k^2+(h-a)^2$. 10. Both the diameters must *meet* the curve; it will be found that this requires the conjugate axis to be *greater* than the transverse axis.

CHAPTER XIII.

1. The equation may be written $(x-2y)(x-2y-2a)=0$, and therefore represents two *parallel* straight lines; a line parallel to them, and midway between them, will be a line of centres. 2. $h=\frac{b}{3}$, $k=\frac{c}{3}$. 3. Two parallel straight lines. 4. A parabola. 5. An hyperbola if the angle A is less than $\frac{\pi}{2}$, an ellipse if it is greater than $\frac{\pi}{2}$, a straight line if it is equal to $\frac{\pi}{2}$. 6. The equation to the hyperbola is $a^2y^2=a^2b^2-4ab^2x+3b^2x^2$; the asymptotes are determined by the equations $ay=\pm\left(x-\frac{2a}{3}\right)b\sqrt{3}$. 8. The locus is then a straight line which coincides with the equal axes. 10. Use Art. 205. 11. $\frac{a\sqrt{2}}{4}$. 13. $\text{Tan}^{-1}\frac{1}{b}$. 14. $\{ay+x\sqrt{(\beta\beta')}\}^2-2a\beta\beta'x-a^2(\beta+\beta')y+a^2\beta\beta'=0$.

17. (1) A circle about the other focus of the given ellipse as centre; (2) an ellipse about the other focus of the given ellipse as focus, and having the same excentricity as the given ellipse. 18. The equation is $(y-3x+1)(y-2x+4)=0$, and therefore represents two straight lines. 24. Use the result given in the last example to Chap. VIII. 26. The equation may be written

$$(x^2+y^2+xy\sqrt{2}-a^2)(x^2+y^2-xy\sqrt{2}-a^2)=0.$$

CHAPTER XIV.

2. Each locus is an ellipse. 4, 5, 6. Use the equation in Art. 294. 7. The equation to the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; the equation to the chord of contact is $\frac{xh}{a^2}+\frac{yk}{b^2}=1$; hence the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{xh}{a^2}+\frac{yk}{b^2}$ represents some locus passing through the points of contact. 10. The equation to the hyperbola

is $(y-k)\,b^2x=(x-h)\,a^2y$. 12. Let y', y'' denote the two ordinates which correspond to the same abscissa x'; then

$$y'=-bx'+\sqrt{(b^2x'^2-ax'^2+f)}, \qquad y''=-bx'-\sqrt{(b^2x'^2-ax'^2+f)}.$$

The equations to the normals are, by Art. 284,

$$(y-y')\,(ax'+by')=(y'+bx')\,(x-x'), \text{ and}$$
$$(y-y'')\,(ax'+by'')=(y''+bx')\,(x-x');$$
$$\text{by addition } (a-b^2)\,x'\,(y+2bx')+bf=0 \;\ldots\; (1);$$
$$\text{by subtraction } b\,(y+bx')-(a-b^2)\,x'=x-x',$$
$$\text{therefore } x'\,(1+2b^2-a)=x-by \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Substitute the value of x' from (2) in (1) and the required equation will be obtained. The locus is an hyperbola. 13. Locus a conic section, which passes through H and R, and through the intersection of the fixed lines. 18. A circle having its centre on the line joining the two points. 19. Two loci, an ellipse, and a parabola. 20. A circle. 23. See Art. 293. 26. Use the equation to the parabola given in Art. 294, and the equation to the circle given in Example 21 to Chap. VI. 29. $r \sin 2\theta = c$. 30. $x^{\frac{4}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}x^{\frac{2}{3}}=a^2$. 32. See Example 30 to Chap. X. 35. An ellipse. 37. In the first case the locus is a circle; in the second it is a straight line. 38. A circle having its centre at H. 44. $\dfrac{b^2x^2}{a^4}+\dfrac{y^2}{b^2}=1$. 46. The equation is $y^2=4a\,(x-8a)$. 50. The line $\dfrac{x}{a}-\dfrac{y}{b}=0$, bisects the chord of contact, and is therefore parallel to the axis of the parabola; if through the point $(a, 0)$ a line be drawn making the same angle with the tangent at that point as the axis makes, the focus must be in this line; $y\,(a+2b\cos\omega)+b\,(x-a)=0$ is the equation to this line. Similarly we can draw a line through the point $(0, b)$ which will also contain the focus. 52. We may take for the equation to one normal $y=mx-am-am^3$, and for the other $x=m'y-am'-am'^3$; also $m'=-m$. Then by addition $y+x=m(x-y)$. Substitute for m in the first equation and reduce; thus we obtain $2a\,(x+y)=(x-y)^2$. 53. We have to eliminate m between

$$y-mx=-\frac{m\,(a^2-b^2)}{\sqrt{(a^2+m^2b^2)}}, \text{ and } my+x=\frac{m\,(a^2-b^2)}{\sqrt{(m^2a^2+b^2)}}.$$

Square and add; we shall obtain after reduction

$$y^2+x^2=\frac{(a^2+b^2)(a^2-b^2)^2}{a^2b^2\left(m-\dfrac{1}{m}\right)^2+(a^2+b^2)^2}\quad\text{.(1).}$$

Also $(y-mx)^2(a^2+m^2b^2)=(my+x)^2(m^2a^2+b^2)$;
by reduction we obtain

$$(a^2y^2-b^2x^2)\left(m-\frac{1}{m}\right)=-2xy(a^2+b^2)\ldots\ldots\ldots\ldots(2).$$

From (1) and (2)

$$(a^2+b^2)(x^2+y^2)(a^2y^2+b^2x^2)^2=(a^2-b^2)^2(a^2y^2-b^2x^2)^2.$$

54. Suppose the figure in Art. 192 to represent the ellipse and the conjugate diameters. Take the equation in Example 23 of Chapter IX. for the equation to the normal at P, and an analogous equation for the normal at D. Let Q denote the point of intersection of these normals, and x, y its co-ordinates. Then it will be found that

$$ax=(a^2-b^2)\sin\phi\cos\phi(\sin\phi-\cos\phi),$$
$$by=(b^2-a^2)\sin\phi\cos\phi(\sin\phi+\cos\phi).$$

Similarly we can determine the co-ordinates of the point of intersection of the normals at P and D; denote this point by R. Then express the area of the triangle CPR, which is one-fourth of the required area.

55. Take the centre of the square as the origin, and the axes parallel to the sides of the square. Then for the equation to the circle take $x^2+y^2=2a^2$, and for the equation to the conic take $y^2-a^2=\lambda(x^2-a^2)$. The equation to the tangent to the circle at the point (x_1, y_1) is $xx_1+yy_1=2a^2$. The equation to the tangent to the conic at the point (x', y') is $yy'-\lambda xx'=a^2(1-\lambda)$. These equations must represent the same line. Hence eliminating λ and x_1 and y_1 we shall arrive at an equation which determines the required locus. It will be found that this equation may be written

$$\{(x'^2+y'^2-2a^2)\}\{a^2(x'^2+y'^2)-2x'^2y'^2\}=0.$$

56. The former part follows from Art. 288. For the latter part proceed thus. Let a perpendicular be drawn from H on

the tangent TQ, and let R denote the intersection of this perpendicular with SQ produced. Then $SR = SQ + QR = 2a$: and $TR = TH$. We have to find the value of the perpendicular from T on SR; denote it by r; then $r2a$ = twice the area of the triangle TSR. Let $TS = c_1$, and TR or $TH = c_2$; then by using the known expression for the area of a triangle in terms of its sides, we have $4ra = \sqrt{(2c_1^2c_2^2 + 8a^2c_1^2 + 8a^2c_2^2 - c_1^4 - c_2^4 - 16a^4)}$. This will lead to the required result. Or thus. Let ϕ denote the angle between HP and TP; then we shall have $r = TP \sin\phi = TP \times \frac{b}{CD}$, where CD is conjugate to CP; see Arts. 181 and 193. And it may be shewn by Art. 208 that

$$\left(\frac{TP}{CD}\right)^2 \qquad + \frac{y^2}{b^2} - 1.$$

CHAPTER XV.

6. $\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma} = 0$. 10. The equation to the conic section being $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$, that to $A'B$ is $(m+n)\,\alpha + l\gamma = 0$, that to $A'C$ is $(m+n)\,\alpha + l\beta = 0$, and that to $A'B'$ is $(m+n)\,\alpha + (l+n)\,\beta - n\gamma = 0$. 13. $lmn + 1 = 0$.

21. $\frac{l}{\lambda} + \frac{m}{\mu} + \frac{n}{\nu} = 0$. 24. Suppose the focus S is to lie on the line $l\alpha + m\beta + n\gamma = 0$. Let α', β', γ' denote the values of α, β, γ respectively for the other focus H of one of the ellipses. Then, by Art. 181, $\alpha\alpha' = \beta\beta' = \gamma\gamma'$ = the square of the semi-axis minor. Hence, substituting in the given equation we obtain $\frac{l}{\alpha'} + \frac{m}{\beta'} + \frac{n}{\gamma'} = 0$, that is, $l\beta'\gamma' + m\gamma'\alpha' + n\alpha'\beta' = 0$. This shews that the locus of H is a conic section passing through the angular points of the triangle.

25. It will be found that the conic sections may be represented by the equations

(1) $\beta\gamma - \alpha^2 = 0$, (2) $\gamma\alpha - \beta^2 = 0$, (3) $\alpha\beta - \gamma^2 = 0$.

Now, (1) may be written $\beta\,(\gamma + \beta - 2\alpha) - (\alpha - \beta)^2 = 0$,

(2) .... $\gamma\,(\alpha + \gamma - 2\beta) - (\beta - \gamma)^2 = 0$,

(3) .... $\alpha\,(\beta + \alpha - 2\gamma) - (\gamma - \alpha)^2 = 0$;

this shews that the tangents to the conic sections at the common point are given by

$$\gamma + \beta - 2\alpha = 0, \qquad \alpha + \gamma - 2\beta = 0, \qquad \beta + \alpha - 2\gamma = 0;$$

these three lines intersect respectively the lines $\alpha = 0$, $\beta = 0$, $\gamma = 0$, in three points which all lie in the line $\alpha + \beta + \gamma = 0$. Again, (1) may be written $\beta(\gamma + 4\alpha + 4\beta) - (\alpha + 2\beta)^2 = 0$, and (2) may be written $\alpha(\gamma + 4\alpha + 4\beta) - (\beta + 2\alpha)^2 = 0$; and this shews that $\gamma + 4\alpha + 4\beta = 0$ is a common tangent of (1) and (2), and this common tangent meets $\gamma = 0$ at the point where $\beta + \alpha - 2\gamma = 0$ meets it. And so on.

26. The equation to the first hyperbola is $\beta\gamma = AA'^2 \sin^2 \frac{A}{2}$; similarly for the others.

27. See Art. 274.

28 and 29. These may be solved by taking oblique axes coinciding with the sides of the triangle. For instance, consider 29. We have $a\alpha + b\beta + c\gamma = -ab \sin C$. Thus the equation may be written $cn\alpha\beta - (l\beta + m\alpha)(ab \sin C + a\alpha + b\beta) = 0$; and taking CA for the axis of x, and CB for the axis of y, we have $\alpha = x \sin C$, $\beta = y \sin C$. Substitute for α and β and then to the equation in x and y we may apply the ordinary test; see Arts. 272 and 278.

30. $\frac{\lambda^2}{l} + \frac{\mu^2}{m} + \frac{\nu^2}{n} = 0.$

31. $u(mn' - m'n) = v(nl' - n'l) = w(lm' - l'm).$

32. Let $S_1 = 0$ be the equation to the inscribed circle, $S_2 = 0$ the equation to the circumscribed circle, these equations not being necessarily in their simplest forms; see Art. 110. Then, if k be a suitable constant, $S_1 - kS_2 = 0$ will represent the *line* required. In this way we shall have

$$\alpha^2 \cos^4 \frac{A}{2} + \beta^2 \cos^4 \frac{B}{2} + \gamma^2 \cos^4 \frac{C}{2} - 2\beta\gamma \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}$$
$$- 2\gamma\alpha \cos^2 \frac{C}{2} \cos^2 \frac{A}{2} - 2\alpha\beta \cos^2 \frac{A}{2} \cos^2 \frac{B}{2}$$
$$- k(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C)$$
$$= (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma),$$

where l, m, n are to be found. Then by comparing like terms we can find l, m, n.

33. It may be shewn that the equation

$$\frac{n\beta+m\gamma}{a}=\frac{l\gamma+n\alpha}{b}$$

represents a diameter; for this equation represents a line passing through the intersection of the tangents at A and B, and through the middle point of AB. Hence the centre of the conic section is determined by

$$\frac{n\beta+m\gamma}{a}=\frac{l\gamma+n\alpha}{b}=\frac{m\alpha+l\beta}{c};$$

and then the required equation can be found. It is

$$\frac{\beta}{m(al-bm+cn)}=\frac{\gamma}{n(al+bm-cn)}.$$

34. Assume for the required equation $\gamma=$ constant, that is $\gamma=k(a\alpha+b\beta+c\gamma)$. Then by applying the result of Example 21 we shall obtain for the required equation $(lb+ma)(a\alpha+b\beta)-nab\gamma=0$.

35. The equation to the conic section may be taken to be $\alpha\beta=k\gamma^2$; and the equation to the line PQ will be $\alpha-\beta=0$. The equation to the chord will be $\alpha-\beta=k'\gamma$. Thus $k(\alpha-\beta)^2=k'^2\alpha\beta$ will represent the lines joining P with the points of intersection of the chord and the conic section. From the symmetrical form of the last equation we infer that one line makes the same angle with the line $\alpha=0$ which the other makes with the line $\beta=0$.

THE END.

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